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Editor : Anil Ahlawat

editorial

Astronomy and Extensions

Astronomy is one of the oldest branches of Science. In India, ancient and modern astronomy is closely related to the religious practices throughout India. Phases of the moon, the effect of stars and the power of the various planets in a particular position of the earth related to the Sun are all elaborately studied. The study of astronomy has become very important because, apart from visual observations and powerful telescopes, the satellites are playing a very important role.

The effect of the various constellations are studied in depth. The composition of the satellites of the Sun such as Moon, Mars and other planets are scanned closely and for some samples were also taken. In applied astronomy, *viz.* astrology, general predictions for the climatic condition of the earth and harvests are made. Apart from these, the orbit of the earth is divided into 12 parts (Rashi's) and according to the time of birth of the person and place of birth, depending on the Rashi, general conclusions are given regarding education, career, marriage and so on.

Studies in Berlin have recently shown that persons born in spring and summer may be great optimists and those born in autumn are less likely to be depressed than winter borns. The scientists from Budapest have shown the same effect from Biochemical studies. While these studies have touched the fringe, our science of astrology has gone deep into the effect of the time and place of birth on humans.

The advice to the students is this. It is in your hand to succeed in life or not. Faith in God, hard work and a smiling face will take you far ahead than blaming the luck.

Anil Ahlawat
Editor

MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 144

JEE MAIN

1. Let m be the number of numbers from the set {1, 2, 3, ..., 2014} which can be expressed as difference of squares of two non negative integers. The sum of the digits of m is

(a) 4 (b) 5 (c) 6 (d) 7

2. $ABCD$ is a parallelogram. M is a point on AB so that $AM = m \cdot AB$ and N is a point on AD so that $AN = n \cdot AD$. If the lines MN and AC meet at the point P, then $\frac{AP}{AC} =$

(a) $\frac{m+n}{2}$ (b) $\sqrt{mn}$ (c) $\frac{mn}{m+n}$ (d) $\frac{2mn}{m+n}$

3. The sets $A = \{z : z^{18} = 1\}$ and $B = \{\omega : \omega^{48} = 1\}$ both are sets of complex roots of unity. The set $C = \{z\omega : z \in A \text{ and } \omega \in B\}$ is also a set of complex roots of unity. The sum of the digits of the number of the distinct elements in C is

(a) 7 (b) 8 (c) 9 (d) 10

4. Let n be the number of 5-letter words using the letters of the word NARENDRA. The sum of the digits of n is

(a) 3 (b) 4 (c) 5 (d) 6

5. The solution curves of the differential equation $\frac{dy}{dx} = \frac{\sqrt{x^2+y^2}-x}{y}$ are

(a) circles (b) parabolas
(c) ellipses (d) hyperbolas

JEE ADVANCED

6. If $A_k = \frac{k(k-1)}{2}\cos\frac{k(k-1)\pi}{2}$ and $N = \sum_{k=1}^{2015} A_k$ then N is divisible by

(a) 2 (b) 3 (c) 5 (d) 7

COMPREHENSION

A hexagon $ABCDEF$ is inscribed in a circle of radius r. Let $AB = 31$, $BC = CD = DE = EF = FA = 81$

7. $r =$

(a) $\frac{243}{\sqrt{11}}$ (b) $\frac{162}{\sqrt{16}}$ (c) $\frac{81}{\sqrt{11}}$ (d) $8\sqrt{11}$

8. $BE =$

(a) 105 (b) 135 (c) 144 (d) 145

INTEGER MATCH

9. The sequence (a_n) satisfies $a_1 = 1$ and $5^{(a_{n+1}-a_n)} = 1 + \frac{1}{n+\frac{2}{3}}$. The least integer value of a_n is

MATCHING LIST

10. Let the range of $f(x)$ be $[a, b]$

	$f(x)$		$b-a$
P.	$\frac{1}{\pi^2}$	1.	$((\cos^{-1}x)^2 - (\sin^{-1}x)^2)$
Q.	$\frac{1}{\pi^2}$	2.	$((\cos^{-1}x)^2 + (\sin^{-1}x)^2)$
R.	$\frac{1}{\pi^3}$	3.	$\frac{1}{\pi^3}((\cos^{-1}x)^3 - (\sin^{-1}x)^3)$
S.	$\frac{1}{\pi^3}$	4.	$((\cos^{-1}x)^3 + (\sin^{-1}x)^3)$

	P	Q	R	S
(a)	2	3	4	1
(b)	3	4	1	2
(c)	4	1	2	3
(d)	1	2	3	4

See Solution set of Maths Musing 143 on page no. 21.

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

PRACTICE PAPER 2015

JEE Main & Advanced

SECTION – I

Straight Objective Type

This section contains 26 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE choice is correct.

1. Which of the following function is injective map?

   (a) $f(x)=x^2+2,\ x\in(-\infty,\infty)$

   (b) $f(x)=|x+2|,\ x\in[-2,\infty)$

   (c) $f(x)=(x-4)(x-5),\ x\in(-\infty,\infty)$

   (d) $f(x)=\dfrac{4x^2+3x-5}{4+3x-5x^2},\ x\in(-\infty,\infty)$

2. Let $f(x)=\dfrac{1}{\sqrt{|x-1|-[x]}}$ where [.] denote the greatest integer function, then domain of $f(x)$ is

   (a) $(-1, 1)$ (b) $(-\infty, 1)$
   (c) $(-\infty, -1)$ (d) none of these

3. If $x_1, x_2, x_3, ..., x_n$ are the roots of the equation, $x^n + ax + b = 0$, then the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) .... (x_1 - x_n)$ is equal to

   (a) $nx_1^{n-1} + a$ (b) $n(x_1)^{n-1}$
   (c) $nx_1 + b$ (d) $nx_1^{n-1} + b$

4. Matrix A is such that $A^2 = 2A - I$, where I is unit matrix, then for $N \geq 2$, A^n is equal to

   (a) $nA - (n - 1)I$ (b) $nA - I$
   (c) $2^{n-1}A - (n - 1)I$ (d) $2^{n-1}A - I$

5. Foot of perpendicular drawn from origin to the plane passing through (1, 0, 0), (0, 1, 0) and (0, 0, 1) is

   (a) (3, 3, 3) (b) $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$
   (c) $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ (d) (2, 2, 2)

6. $\vec{a},\vec{b},\vec{c}$ are three vectors so that each one is perpendicular to the sum of the other two and also $|\vec{a}|=5, |\vec{b}|=6, |\vec{c}|=7$. Then $\left|\vec{a}+\vec{b}+\vec{c}\right|$ is

   (a) $2\sqrt{55}$ (b) $\sqrt{110}$
   (c) $\sqrt{55}$ (d) none of these

7. The condition to the imposed on β so that $(0, \beta)$ lies on or inside the triangle having sides $y = 3x + 2 = 0$, $3y - 2x - 5 = 0$ and $4y + x - 14 = 0$ is

   (a) $0<\beta<\frac{5}{3}$ (b) $0<\beta<\frac{7}{2}$
   (c) $\frac{5}{3}\leq\beta\leq\frac{7}{2}$ (d) $\frac{5}{2}\leq\beta\leq\frac{7}{2}$

8. If $a, b, c, d > 0$; $x \in R$ and $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \leq 0$, then $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$ is equal to

   (a) 1 (b) −1
   (c) 0 (d) 2

By : S.K. Sir, Heritage IIT Academy, Ph : 0755-4074100.

9. $\int_a^b f(x)\,dx$ is equal to

(a) $\int_{a+t}^{b+t} f(x)\,dx$ (b) $\int_{a-t}^{b-t} f(x-t)\,dx$

(c) $\int_{a-t}^{b-t} f(x+t)\,dx$ (d) none of these

10. The mean deviation of an ungrouped data is 10. If each observation is increased by 4%, then the revised mean deviation is

(a) 10.4 (b) 10.04

(c) 9.6 (d) 10.0

11. If A and B are square matrices of the same order, then which of the following is always true?

(a) $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$

(b) A and B are non-zero and $|AB| = 0 \Leftrightarrow |A| = 0$ and $|B| = 0$

(c) $(AB)^{-1} = A^{-1}B^{-1}$

(d) $(A + B)^{-1} = A^{-1} + B^{-1}$

12. Two variable chords AB and BC of a circle $x^2 + y^2 = r^2$ are such that $AB = BC = r$. Then the locus of point of intersection of tangents at A and C is

(a) $x^2 + y^2 = 2r^2$ (b) $x^2 + y^2 = 3r^2$

(c) $x^2 + y^2 = 4r^2$ (d) $x^2 + y^2 = 5r^2$

13. Consider the equation $x^4 + x^2 + 1 = 0$. If x_1, x_2, x_3, x_4 are roots of this equation, then value of $x_1^6 + x_2^6 + x_3^6 + x_4^6$ equals

(a) 1 (b) 2

(c) 3 (d) 4

14. If z lies on the circle $|z - 1| = 1$, then $\frac{z-2}{z}$ equals

(a) 0 (b) 2

(c) -1 (d) none of these

15. One hundred identical coins, each with probability p of showing heads are tossed once. If $0 < p < 1$and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, the value of p is

(a) $\frac{1}{2}$ (b) $\frac{51}{101}$

(c) $\frac{49}{101}$ (d) none of these

16. On the portion of the straight line $x + y = 12$ which is intercepted between the axes, a square is constructed, away from the origin, with this portion as one of its side. If p denotes the perpendicular distance of a side of this square from the origin, then the maximum value of p is

(a) $2\sqrt{3}$ (b) $3\sqrt{2}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{3}{\sqrt{2}}$

17. If P is a point on the rectangular hyperbola $x^2 - y^2 = a^2$, C is its centre and S, S' are the two foci, then $SP \cdot S'P$ is equal to

(a) 2 (b) $(CP)^2$ (c) $(CS)^2$ (d) $(SS')^2$

18. The angle between the lines whose direction cosines are given by the equations $l^2 + m^2 - n^2 = 0$, $l + m + n = 0$ is

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/2$

19. Let $f(x) = \frac{|x|}{x}$ if $x \neq 0$ and $f(0) = 0$ and $a, b \in R$ be such that $a < b$, then value of $I = \int_a^b f(x)\,dx$ is

(a) $|b| - |a|$ (b) $\frac{1}{2}(b^2 - a^2)$

(c) $\max\{|a|, |b|\}$ (d) $\min\{|a|, |b|\}$

20. If ω is complex cube root of unity, and $A = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$, then A^{100} is equal to

(a) A (b) $-A$

(c) O (d) none of these

Solution Sender of Maths Musing

SET-143

1. Khokon Kumarnandi (West Bengal)
2. Gouri Sankar Adhikari, Rintu Giri (P. Medinipur)
3. R. Akhil, Soham, Snigdha, Hansica, sikhira (Hyderabad)
4. Samay Kathe (Maharashtra)
5. Rushikesh Joshi (Maharashtra)
6. Khwaja Sami Baig (Azamgarh)
7. Harshita, Komal, Sunandini, Avinash, N. Jayanthi (Hyderabad)

21. The value of the determinant $\begin{vmatrix} ka & k^2+a^2 & 1 \\ kb & k^2+b^2 & 1 \\ kc & k^2+c^2 & 1 \end{vmatrix}$ is

(a) $k(a+b)(b+c)(c+a)$

(b) $kabc(a^2+b^2+c^2)$

(c) $k(a-b)(b-c)(c-a)$

(d) $k(a+b-c)(b+c-a)(c+a-b)$

22. If in the expansion of $\left(x^3-\frac{1}{x^2}\right)^n, n\in N,$ sum of the coefficient of x^5 and x^{10} is 0, then value of n is

(a) 5 (b) 10

(c) 15 (d) none of these

23. If A, B and C are three events such that $P(B)=\frac{3}{4}$, $P(A\cap B\cap C')=\frac{1}{3}$ and $P(A'\cap B\cap C')=\frac{1}{3}$, then $P(B\cap C)$ is equal to

(a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{15}$ (d) $\frac{1}{9}$

24. The value of $\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)+\cos(\tan^{-1}2\sqrt{2})$ is

(a) $\frac{12}{13}$ (b) $\frac{13}{14}$

(c) $\frac{14}{15}$ (d) none of these

25. If the lines $x + 3y = 4$ and $3x + y = 4$ cuts the coordinate axes at four concyclic points, then the radius of that circle is equal to

(a) $\frac{4\sqrt{5}}{3}$ (b) $\frac{3\sqrt{5}}{4}$ (c) $\frac{4}{3}$ (d) $\frac{\sqrt{5}}{3}$

26. If the coordinates axes are rotated by an angle 45° in clockwise direction (keeping origin fixed), then equation of hyperbola $x^2 - y^2 = 2$ becomes

(a) $xy - 1 = 0$ (b) $xy + 1 = 0$

(c) $xy + 2 = 0$ (d) none of these

SECTION-II

Multiple Correct Choice Type

This section contains 4 multiple correct choice type questions. Each question has 4 choices: (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

27. Orthogonal trajectories of the system of curves $\left(\frac{dy}{dx}\right)^2=\frac{a}{x}$ are

(a) $9a(y + c)^2 = 4x^3$ (b) $y+c=\frac{-2}{3\sqrt{a}}x^{3/2}$

(c) $y+c=\frac{2}{3\sqrt{a}}x^{3/2}$ (d) $9a(y - c)^2 = 5x^3$

28. If $\frac{{}^nC_1}{{}^nC_0}+2\frac{{}^nC_2}{{}^nC_1}+3\frac{{}^nC_3}{{}^nC_2}+......+n\frac{{}^nC_n}{{}^nC_{n-1}}=S$, then S will be

(a) an integer (b) $\frac{1}{2}n(n-1)$

(c) $\frac{1}{2}n(n+1)$ (d) 55, if $n = 10$

29. If a, b, c, d, e are positive real numbers such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, then

(a) $1 \le e < 2$ (b) $0<e\le\frac{16}{5}$

(c) $2 < e < 3$ (d) $0 < e < 4$

30. $\sin\theta+\sqrt{3}\cos\theta=6x-x^2-11,\ 0\le\theta\le4\pi, x\in R$ holds for

(a) no value of x and θ

(b) one value of x and two values of θ

(c) two values of x and two values of θ

(d) two pair of values of (x, θ)

SOLUTIONS

1. (b) 2. (b)

3. (a) : $x^n + ax + b = (x - x_1)(x - x_2) \,....\, (x - x_n)$

$$\Rightarrow (x-x_2)(x-x_3)........(x-x_n)=\frac{x^n+ax+b}{x-x_1}$$

$$=\lim_{x\to x_1}\frac{x^n+ax+b}{x-x_1}=nx_1^{n-1}+a$$

4. (a) : $A^2 = 2A - I$ ...(i)

Multiplying by A, we have

$$A^3 = 2A^2 - A = 2(2A - I) - A \Rightarrow A^3 = 3A - 2I$$

Again multiplying by A

$$A^4 = 3A^2 - 2AI \Rightarrow A^4 = 3(2A - I) - 2A$$

$$\Rightarrow A^4 = 6A - 3I - 2A \Rightarrow A^4 = 4A - 3I$$

Hence by induction $A^n = nA - (n - 1)I$

5. (c)

6. (b) : $\vec{a}\cdot(\vec{b}+\vec{c})+\vec{b}\cdot(\vec{c}+\vec{a})+\vec{c}\cdot(\vec{a}+\vec{b})=0$

$$\Rightarrow \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=0$$

$$|\vec{a}+\vec{b}+\vec{c}|^2=\sum\vec{a}^2+2\sum\vec{a}\cdot\vec{b}=110$$

7. (c)

8. (c) : $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \le 0$

$$\Rightarrow (ax - b)^2 + (bx - c)^2 + (cx - d)^2 \le 0$$

$$\Rightarrow (ax - b)^2 + (bx - c)^2 + (cx - d)^2 = 0$$

$$\Rightarrow \frac{b}{a}=\frac{c}{b}=\frac{d}{c}=x \Rightarrow b^2=ac$$

or, $2\log b = \log a + \log c$

Now, $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = \begin{vmatrix} 130 & 54 & \log a+\log c \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$

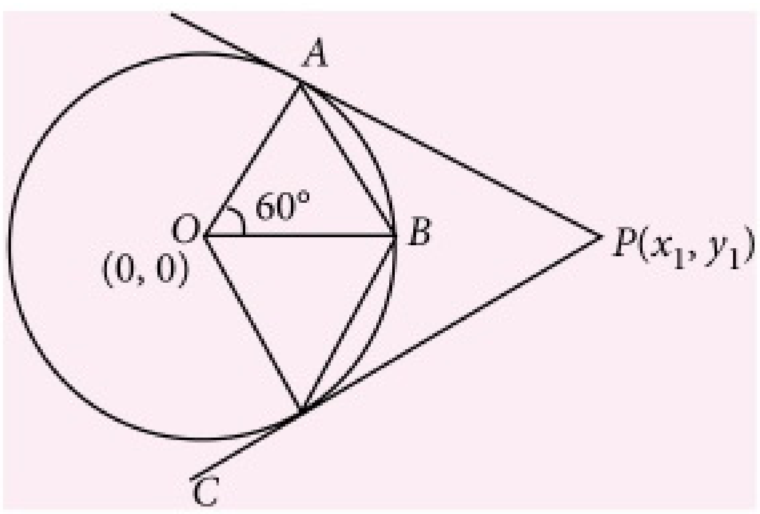
Applying $R_1 \to R_1 + R_3$

$$=2\begin{vmatrix} 65 & 27 & \log b \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}=0$$

9. (c) **10. (a)** **11. (a)**

12. (c) : $\angle AOB = 60°$

$$\frac{PA}{r}=\tan 60°=\sqrt{3} \Rightarrow PA=\sqrt{3}r$$



$$\sqrt{x_1^2+y_1^2-r^2}=\sqrt{3}r \Rightarrow x_1^2+y_1^2=4r^2$$

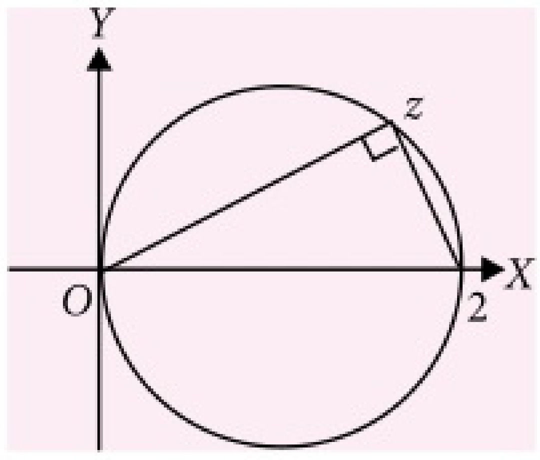
$\Rightarrow$ Locus of $P(x_1, y_1)$ is $x^2 + y^2 = 4r^2$

13. (d) : $x^2=\dfrac{-1\pm i\sqrt{3}}{2}=\omega,\omega^2$

$x^6 = \omega^6, (\omega^2)^6$ *i.e.* $x^6 = 1$

$$\Rightarrow x_1^6+x_2^6+x_3^6+x_4^6=4$$

14. (d) : Note that $|z - 1| = 1$ represents a circle with the segment joining $z = 0$ and $z = 2 + 0i$ as a diameter.



If z lies on the circle, then $\arg\left(\dfrac{z-2}{z-0}\right)=\dfrac{\pi}{2} \Rightarrow \dfrac{z-2}{z}$ is purely imaginary.

15. (b) : Let X be the number of coins showing heads. Then X follows a binomial distribution with parameters $n = 100$ and p.

Since $P(X = 50) = P(X = 51)$, we get

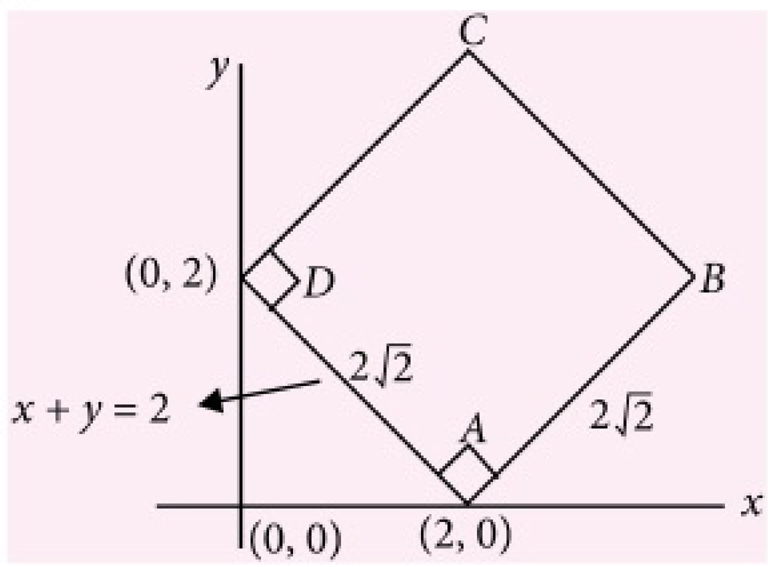
$$^{100}C_{50}p^{50}(1-p)^{50}={}^{100}C_{51}p^{51}(1-p)^{49}$$

$$\Rightarrow \frac{100!}{50!50!}\cdot\frac{51!49!}{100!}=\frac{p}{1-p}$$

$$\Rightarrow \frac{51}{50}=\frac{p}{1-p} \Rightarrow 51-51p=50p \Rightarrow p=\frac{51}{101}$$

16. (b) : Clearly maximum value of p = perpendicular distance from (0, 0) to AD + side of the square

$$=\frac{2}{\sqrt{2}}+2\sqrt{2}=3\sqrt{2}$$



17. (b) : By definition of the hyperbola

$SP = e$(distance of P from $x=a/\sqrt{2}$

$$=\sqrt{2}\,|x-(a/\sqrt{2})|)$$

Similarly, $S'P=\sqrt{2}\,|x+(a/\sqrt{2})$

So that $SP\cdot SP'=2|x^2-(a^2/2)|=2x^2-a^2$

$$= x^2 + y^2 = (CP)^2$$

($\because$ P lies on the hyperbola $x^2 - y^2 = a^2$)

18. (c)

19. (a) : Note that $f(x)=\begin{cases}-1, & \text{if} \quad x<0\\ 0, & \text{if} \quad x=0\\ 1, & \text{if} \quad x>0\end{cases}$

If $0 \le a < b$, then $I=\int_a^b dx = b-a=|b|-|a|$

If $a < b < 0$, then

$$I=\int_a^b(-1)dx=-b+a=|b|-|a|.$$

If $a < 0 < b$, then

$$I=\int_a^b f(x)dx=\int_a^0 f(x)dx+\int_0^b f(x)dx=|b|-|a|$$

20. (a) : We have

$$A^2=\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}=\begin{bmatrix}\omega^2 & 0\\ 0 & \omega^2\end{bmatrix},$$

and $A^3=A^2A=\begin{bmatrix}\omega^2 & 0\\ 0 & \omega^2\end{bmatrix}\begin{bmatrix}\omega & 0\\ 0 & \omega\end{bmatrix}$

$$=\begin{bmatrix}\omega^3 & 0\\ 0 & \omega^3\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=I \qquad [\because \omega^3=1]$$

Now, $A^{100}=A^{99}A=(A^3)^{33}A=I^{33}A=A.$

21. (c) **22. (c)**

23. (a) : We have

$P(B\cap C')=P[(A\cup A')\cap(B\cap C')]$

$=P(A\cap B\cap C')+P(A'\cap B\cap C')=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$

Now, $P(B\cap C)=P(B)-P(B\cap C')=\frac{3}{4}-\frac{2}{3}=\frac{1}{12}$

24. (c) : Let $\tan^{-1}\frac{1}{3}=\alpha$ and $\tan^{-1}2\sqrt{2}=\beta$. Then

$\tan\alpha=\frac{1}{3}$ and $\tan\beta=2\sqrt{2}$, so that

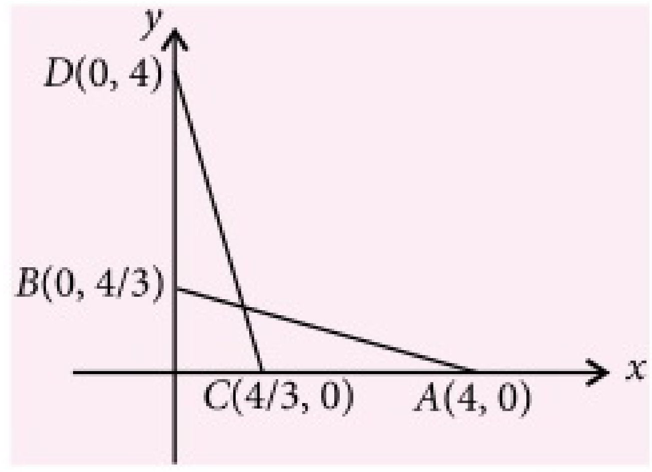
$$\sin\left(2\tan^{-1}\left(\frac{1}{3}\right)\right)+\cos(\tan^{-1}2\sqrt{2})=\sin 2\alpha+\cos\beta$$

$$=\frac{2\tan\alpha}{1+\tan^2\alpha}+\frac{1}{\sqrt{1+\tan^2\beta}} \quad \left[\because -\frac{\pi}{2}<\beta<\frac{\pi}{2}\right]$$

$$=\frac{2\left(\frac{1}{3}\right)}{1+\left(\frac{1}{9}\right)}+\frac{1}{\sqrt{1+8}}=\frac{2}{3}.\frac{9}{10}+\frac{1}{3}=\frac{3}{5}+\frac{1}{3}=\frac{14}{15}$$

25. (a) : $AB : x + 3y = 4$

$CD : 3x + y = 4$



Co-ordinates of centre $=\left(\frac{8}{3}, \frac{8}{3}\right)$

Radius $=\sqrt{\left(\frac{8}{3}-\frac{4}{3}\right)^2+\left(\frac{8}{3}-0\right)^2}=\frac{4\sqrt{5}}{3}$

26. (a)

27. (a, b, c) : Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, we get

$$\left(\frac{dy}{dx}\right)^2=\frac{x}{a}$$

$$dy=\pm\frac{\sqrt{x}}{\sqrt{a}}dx \;\Rightarrow\; y=\pm\frac{1}{\sqrt{a}}\frac{2}{3}x^{3/2}-c$$

$$\Rightarrow\; (y+c)^2=\frac{4}{9a}x^3 \;\Rightarrow\; 9a(y+c)^2=4x^3$$

28. (a, c, d)

29. (a, b, c) : Given

$$\left(\frac{a+b+c+d}{4}\right)^2\le\frac{a^2+b^2+c^2+d^2}{4}$$

$$\Rightarrow\; \left(\frac{8-e}{4}\right)^2\le\frac{16-e^2}{4} \;\Rightarrow\; 4-e+\frac{e^2}{16}\le 4-\frac{e^2}{4}$$

$$\Rightarrow\; \frac{e}{16}(5e-16)\le 0. \quad \therefore\; 0<e\le\frac{16}{5}$$

30. (b, d) : $\sin\theta+\sqrt{3}\cos\theta=-2-(x^2-6x+9)$

$= -2 - (x - 3)^2$

$\because\; \sin\theta+\sqrt{3}\cos\theta\le -2$

But minimum value of $\sin\theta+\sqrt{3}\cos\theta=-2$ and then $x = 3$

$\therefore\; x=3$ and $\cos\left(\theta-\frac{\pi}{6}\right)=-1$

$$\theta-\frac{\pi}{6}=\pi, 3\pi;\; \theta=\frac{7\pi}{6}, \frac{19\pi}{6}$$

PRACTICE PAPER 2015

JEE (Main & Advanced) and other PETs

PAPER-1

SECTION-I

MULTIPLE CORRECT CHOICE TYPE

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONE or MORE may be correct.

1. If a perpendicular is drawn from the point (7, 14, 5) to the plane $2x + 4y - z = 2$, then
   (a) length of the perpendicular is $3\sqrt{21}$
   (b) length of the perpendicular is $\sqrt{63}$
   (c) foot of the perpendicular is (1, 2, 8)
   (d) foot of the perpendicular is (2, 4, –1)

2. A direction line makes angles 60° and 45° with the x and y axes respectively. What angle does it make with the axis of z ?
   (a) $\pi/6$ (b) $\pi/3$ (c) $2\pi/3$ (d) $\pi/4$

3. The vector $\vec{A} = 2\hat{i} - \hat{j} + \hat{k}, \vec{B} = \hat{i} - 3\hat{j} - 5\hat{k}$, $\vec{C} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ are
   (a) the vertices of a right angled triangle
   (b) coplanar
   (c) the vertices of an isosceles triangle
   (d) non-coplanar

4. $\cot^{-1}\left(\frac{-3}{4}\right) =$
   (a) $\pi + \sin^{-1}\left(\frac{-4}{5}\right)$ (b) $\pi + \tan^{-1}\left(\frac{-4}{3}\right)$
   (c) $\tan^{-1}\left(\frac{-4}{3}\right)$ (d) $\tan^{-1}\left(\frac{-4}{3}\right) - \pi$

5. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in plane of $\vec{b}$ and $\vec{c}$, whose projection on $\vec{a}$ is of magnitude $\sqrt{2/3}$ is
   (a) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (b) $2\hat{i} + 3\hat{j} + 3\hat{k}$
   (c) $-2\hat{i} - \hat{j} + 5\hat{k}$ (d) $2\hat{i} + \hat{j} + 5\hat{k}$

6. Let $\vec{a}$ and $\vec{b}$ be two non-collinear unit vectors. If $\vec{u} = \vec{a} - (\vec{a}\cdot\vec{b})\vec{b}$ and $\vec{v} = \vec{a} \times \vec{b}$, then $|\vec{v}|$ is
   (a) $|\vec{u}|$ (b) $|\vec{u}| + |\vec{u}\cdot\vec{a}|$
   (c) $|\vec{u}| + |\vec{u}\cdot\vec{b}|$ (d) $|\vec{u}| + \vec{u}\cdot(\vec{a} + \vec{b})$

7. If $\sin\alpha = -3/5$ and α lies in the third quadrant, then the value of $\cos(\alpha/2)$ can be
   (a) 1/5 (b) $-1/\sqrt{10}$
   (c) –1/5 (d) $1/\sqrt{10}$

8. If $\cos x = \sqrt{1 - \sin 2x}, 0 \le x \le \pi$, then x is equal to
   (a) π (b) 2π
   (c) $\tan^{-1}2$ (d) none of these

9. The fundamental period of $\cos(\cos 2x) + \cos(\sin 3x)$ is
   (a) π (b) 2π
   (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

10. The equation(s) of the line of shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ is/are

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(a) $3(x-21) = 3y + 92 = 3z - 32$

(b) $\dfrac{x-(62/3)}{1/3} = \dfrac{y+31}{1/3} = \dfrac{z-(31/3)}{1/3}$

(c) $\dfrac{x-21}{1/3} = \dfrac{y+(92/3)}{1/3} = \dfrac{z-(32/3)}{1/3}$

(d) $\dfrac{x-2}{1/3} = \dfrac{y+3}{1/3} = \dfrac{z-1}{1/3}$

SECTION-II

ONE INTEGER VALUE CORRECT TYPE

This section contains 10 questions. Each question, when worked out will result in one integer from 0 to 9 (both inclusive).

11. The distance of point (3, 8, 2) from the line $\dfrac{x-1}{2} = \dfrac{y-3}{4} = \dfrac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z + 15 = 0$ is

12. The shortest distance between z-axis and the line $x + y + 2z - 3 = 0$, $2x + 3y + 4z - 4 = 0$ is

13. The sum of the roots of the equation

$$4\sin^3\left(\frac{\pi}{2}+x\right) - 4\cos^2 x - \cos(\pi + x) - 1 = 0$$

in the interval $[0, 4\pi]$ is $p\pi$. Find p.

14. In ΔABC, $\angle A = 30°$ and $\angle C = 105°$. Find k such that, $k = \left(\dfrac{c^2 - b^2}{a^2}\right)^2$.

15. If $\sec A = x + \dfrac{1}{4x}$ and $\sec A + \tan A = kx$, then find k.

16. Find θ (in degrees) satisfying the equation, $\tan(18°)\tan(24°)\tan(36°) = \tan\theta$, $(0° < \theta < 45°)$.

17. Let $\vec{a}$ and $\vec{b}$ be two unit vectors. If $\vec{c}$ is a vector such that $\vec{c} + (\vec{c} \times \vec{a}) = \vec{b}$, then the maximum value of $|(\vec{a} \times \vec{b}) \cdot \vec{c}|$ is $A \times 10^{-1}$. Find A.

18. The straight lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular, then the value of $\dfrac{f}{a} + \dfrac{g}{b} + \dfrac{h}{c}$ is

19. If the distance of the point $B(\hat{i} + 2\hat{j} + 3\hat{k})$ from the line which is passing through $A(4\hat{i} + 2\hat{j} + 2\hat{k})$ and which is parallel to the vector $\vec{c} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ is λ, then value of $\lambda^2 - 1$ is

20. If l is the shortest distance between the lines $\dfrac{x-3}{2} = \dfrac{y+15}{-7} = \dfrac{z-9}{5}$ and $\dfrac{x+1}{2} = \dfrac{y-1}{1} = \dfrac{z-9}{-3}$, then determine $[l]\cdot$

($[x]$ denotes G.I.F.)

PAPER-2

SECTION-I

SINGLE CORRECT OPTION

This section contains 10 multiple choice questions. Each question has four choices (a), (b), (c) and (d) for its answer, out of which ONLY ONE is correct.

1. At $x = \dfrac{5\pi}{6}$, $f(x) = 2\sin(3x) + 3\cos(3x)$ is

(a) maximum (b) minimum

(c) zero (d) None of these

2. If a triangle ABC satisfies $(a + b)^2 = c^2 + ab$ and $\sin A + \sin B + \sin C = 1 + (\sqrt{3}/2)$, then the angles of the triangle are

(a) 120°, 30°, 30° (b) 108°, 36°, 36°

(c) 60°, 60°, 60° (d) 30°, 75°, 75°

3. If $\sum_{i=1}^{2n} \sin^{-1}(x_i) = n\pi$, then $\sum_{i=1}^{2n} x_i$ equals

(a) $2n + 1$ (b) $2n$

(c) $\dfrac{n(n+1)}{2}$ (d) $n \times (2n + 1)$

4. If $K\vec{r} + \vec{r} \times \vec{a} = \vec{b}$, where K is a non-zero scalar and $\vec{a}, \vec{b}$ are two given vectors. Then $\vec{r}$ will be

(a) $\dfrac{1}{K^2 + \vec{a}^2}\left(K\vec{b} + \dfrac{\vec{a}\cdot\vec{b}}{K}\vec{a} + \vec{a} \times \vec{b}\right)$

(b) $\frac{1}{K^2+\vec{a}^2}\left(K\vec{b}-\frac{\vec{a}\cdot\vec{b}}{K}\vec{a}+\vec{a}\times\vec{b}\right)$

(c) $\frac{1}{K^2+\vec{a}^2}\left(K\vec{b}-\frac{\vec{a}\cdot\vec{b}}{K}\vec{a}-\vec{a}\times\vec{b}\right)$

(d) none of these

5. If the vectors $a\hat{i}+\hat{j}+\hat{k}, \hat{i}+b\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+c\hat{k} (a \neq b, c \neq 1)$ are coplanar, then the value of $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}$ is

(a) 1 (b) 2 (c) –1 (d) –2

6. The angle between the straight lines

$\frac{x+1}{2}=\frac{y-2}{5}=\frac{z+3}{4}$ and $\frac{x-1}{1}=\frac{y+2}{2}=\frac{z-3}{-3}$ is

(a) 45° (b) 30° (c) 60° (d) 90°

7. The distance of the point where the line $\frac{x+1}{2}=\frac{y+1}{3}=\frac{z+1}{4}$ meets the plane $x + 2y + 3z = 14$, from the origin is

(a) $\sqrt{15}$ (b) $\sqrt{14}$ (c) 7 (d) $\sqrt{7}$

8. If the planes $x = cy + bz$, $y = az + cx$ and $z = bx + ay$, pass through one line, then $a^2 + b^2 + c^2 + 2abc =$

(a) ab (b) 1 (c) bc (d) 0

9. The maximum value of $(\sin \alpha_1)(\sin \alpha_2) \ldots\ldots (\sin \alpha_n)$ under the restrictions $0 \leq \alpha_1, \alpha_2, \ldots\ldots, \alpha_n < \frac{\pi}{2}$ and $(\tan \alpha_1)(\tan \alpha_2) \ldots\ldots (\tan \alpha_n) = 1$ is

(a) $\frac{1}{2^n}$ (b) $\frac{1}{2n}$ (c) $\frac{1}{2^{n/2}}$ (d) 1

10. If k and K are minimum and maximum values of $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ respectively, then

(a) $k=\frac{\pi}{4}, K=\frac{3\pi}{4}$ (b) $k=0, K=\pi$

(c) $k=\pi/2, K=\pi$ (d) not defined

SECTION-II

PARAGRAPH TYPE

This section contains 2 paragraphs. Based upon each of the paragraphs 3 multiple choice questions have to be answered. Each of these questions has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Paragraph for Q. No. 11 to 13

Let $a, b, c, d \in R$. Then the cubic equation of the type $ax^3 + bx^2 + cx + d = 0$ has either one root real or all three roots are real. But in case of trigonometric equations of the type $a \sin^3 x + b \sin^2 x + c \sin x + d = 0$ can possess several solutions depending upon the domain of x. To solve an equation of the type $a\cos\theta + b\sin\theta = c$. The equation can be written as $\cos(\theta - \alpha) = c/\sqrt{(a^2+b^2)}$.

The solution is $\theta = 2n\pi + \alpha \pm \beta$, where $\tan\alpha = b/a$, $\cos\beta = c/\sqrt{(a^2+b^2)}$.

11. On the domain $[-\pi, \pi]$, the equation

$4\sin^3 x + 2\sin^2 x - 2\sin x - 1 = 0$ possess

(a) only one real root

(b) three real roots

(c) four real roots

(d) six real roots

12. In the interval $[-\pi/4, \pi/2]$, the equation;

$\cos 4x + \frac{10\tan x}{1+\tan^2 x} = 3$ has

(a) no solution (b) one solution

(c) two solutions (d) three solutions

13. Let $0 \leq x < 1$. Then the equation

$$3\sin^{-1}\left(\frac{2x}{1+x^2}\right) - 4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) + 2\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{\pi}{3}$$

possesses

(a) no solution (b) one solution

(c) two solutions (d) three solutions

Paragraph for Q. No. 14 to 16

A system of vectors $\vec{a}_1, \vec{a}_2, \ldots., \vec{a}_n$ is said to be linearly dependent if there exists a system of scalars $c_1, c_2, \ldots\ldots, c_n$ (not all zero) such that $c_1\vec{a}_1 + c_1\vec{a}_2 + \ldots\ldots + c_n\vec{a}_n = \vec{0}$

That means $\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n$ are linearly dependent iff one can be expressed as the linear combination of others. Again $\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n$ is said to be linearly independent, if there exists scalars $c_1, c_2, \ldots\ldots, c_n$ such that

$$c_1\vec{a}_1 + c_2\vec{a}_2 + \ldots.. + c_n\vec{a}_n = \vec{0}$$
$$\Rightarrow \; c_1 = c_2 = \ldots. = c_n = 0$$

14. If $\vec{a}, \vec{b}, \vec{c}$ are non-zero non-coplanar vectors, then $\vec{r}_1 = 2\vec{a} - 3\vec{b} + \vec{c}, \vec{r}_2 = 3\vec{a} - 5\vec{b} + 2\vec{c}, \vec{r}_3 = 4\vec{a} - 5\vec{b} + \vec{c}$ are

(a) linearly independent
(b) $\vec{r}_3 = 4\vec{r}_2 - 2\vec{r}_1$
(c) $\vec{r}_3 = 5\vec{r}_1 - 2\vec{r}_2$ (d) none of these

15. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}, \vec{b} = 4\hat{i} + 3\hat{j} + 4\hat{k}$ and $\vec{c} = \hat{i} + \alpha\hat{j} + \beta\hat{k}$ are linearly dependent vectors and $|\vec{c}| = \sqrt{3}$, then

(a) $\alpha = 1, \beta = -1$ (b) $\alpha = 1, \beta = \pm 1$
(c) $\alpha = -1, \beta = \pm 1$ (d) $\alpha = \pm 1, \beta = 1$

16. The vectors $\hat{i} - 3\hat{j} + 2\hat{k}, 2\hat{i} - 4\hat{j} - 4\hat{k}, 3\hat{i} + 2\hat{j} - \hat{k}$ are

(a) linearly independent
(b) linearly dependent
(c) nothing can be said
(d) collinear

SECTION-III

MATCHING LIST TYPE (ONLY ONE OPTION CORRECT)

This section contains 4 questions, each having two matching lists. Choices for the correct combination of elements from List-I and List-II are given as options (a), (b), (c) and (d), out of which ONE is correct.

17. Match the following:

	Column I		Column II
(P)	The number of solutions of the equation $\tan x + \sec x = 2\cos x$ in the interval $[0, 2\pi]$ is	1.	2
(Q)	The number of roots of the equation $x + 2\tan x = \pi/2$ in the interval $[0, 2\pi]$ is	2.	$[\pi]$
(R)	The number of solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \le x < 2\pi$ is	3.	0
(S)	The value of $\tan^{-1}[\pi] + \tan^{-1}[-\pi] + 1 =$	4.	1

	P	Q	R	S
(a)	3	1	4	2
(b)	3	2	1	4
(c)	1	2	4	3
(d)	3	2	4	1

18. Match the following:

	Column I		Column II
(P)	If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then $(\vec{a}+\vec{b}+\vec{c}).[(\vec{a}+\vec{b})\times(\vec{a}+\vec{c})] =$	1.	$\frac{1}{4}a^2b^2$
(Q)	If $\vec{b}$ and $\vec{c}$ are any two non-collinear unit vectors and $\vec{a}$ is any vector, then $(\vec{a}\cdot\vec{b})\vec{b} + (\vec{a}\cdot\vec{c})\vec{c} + \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\lvert\vec{b}\times\vec{c}\rvert}(\vec{b}\times\vec{c}) =$	2.	$-[\vec{a}\,\vec{b}\,\vec{c}]$
(R)	If $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar vectors, then $[\vec{a}+\vec{b}+\vec{c}\;\; \vec{a}-\vec{c}\;\; \vec{a}-\vec{b}] =$	3.	$\vec{a}$
(S)	Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are non-zero vectors s.t. $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ (and angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$), then $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$	4.	$-3[\vec{a}\,\vec{b}\,\vec{c}]$

	P	Q	R	S
(a)	2	1	4	3
(b)	2	3	4	1
(c)	2	1	3	4
(d)	1	2	3	4

Contd. on page no. 93

MATHS MUSING

SOLUTION SET-143

1. (c): $\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{c+x}=\frac{2}{x}$

$\Rightarrow\ x^3 - (ab + bc + ca)\,x - 2abc = 0$

Coeff. of $x^2 = 0 \Rightarrow$ sum of the roots = 0, but ω, ω^2 are roots. Since $1 + \omega + \omega^2 = 0 \Rightarrow$ The other root is 1.

$\therefore\ \Sigma\frac{1}{a+1}=2$

2. (c): $I=2\int_0^{\frac{\pi}{2}}\frac{\sin\left(\left(x-\frac{\pi}{6}\right)-\left(x-\frac{\pi}{3}\right)\right)dx}{\cos\left(x-\frac{\pi}{6}\right)\cos\left(x-\frac{\pi}{3}\right)}$

$=2\int_0^{\frac{\pi}{2}}\left(\tan\left(x-\frac{\pi}{6}\right)-\tan\left(x-\frac{\pi}{3}\right)\right)dx=2\ln 3$

3. (d) : $x^2 = 4y \Rightarrow x = 2t,\ y = t^2$

Normal at $t : x + ty = 2t + t^3$.

It passes through $(1, 2) \Rightarrow 1 = t^3,\ t = 1,\ x + y = 3$.

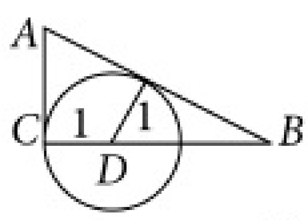
It meets $x^2 = 4y$ at $(-6, 9), (2, 1)$

$\therefore$ Area $=\int_{-6}^{2}\left(3-x-\frac{x^2}{4}\right)dx=\frac{64}{3}$.

4. (c): $x - 2y + 2z = 4,\ x + 3z = 11$ determine a line through the point $(-10, 0, 7)$. Its d.r.s. are $(6, 1, -2)$

The distance of $\vec{c}$ from the line $\vec{x}=\vec{a}+\lambda\vec{b}$ is $\frac{|(\vec{c}-\vec{a})\times\vec{b}|}{|\vec{b}|}=3$, using $\vec{c}=3\hat{i}+5\hat{k},\ \vec{a}=-10\hat{i}+7\hat{k}$, and $\vec{b}=6\hat{i}+\hat{j}-2\hat{k}$.

5. (c): Given $S = 4$



$\Rightarrow\ AB = 8 - (a + b)$

$\Rightarrow\ (8 - (a + b))^2 = a^2 + b^2$

$\Rightarrow\ a(b - 8) = 8(b - 4)$ ...(i)

$\sin B=\frac{1}{a-1}=\frac{b}{8-a-b}\Rightarrow a(b+1)=8$ ...(ii)

(i), (ii) $\Rightarrow b = 2,\ a = \frac{8}{3}$

$\Delta=\frac{1}{2}ab=\frac{8}{3}$

6. (a, c, d): Consider the seven subsets of numbers leaving remainders 0, 1, 2, 3, 4, 5, 6 when divided by 7. The number of subsets $\{x, y\}$ is

$\binom{8}{2}+6\binom{7}{2}+8\times 7+2\times 7^2=308=2^2\cdot 7\cdot 11$

7. (b) : Solving $z^2 - iz - 1 = 0$ gives $z = -i\omega, -i\omega^2$

$z^{2015} = i\omega^2, i\omega,\ \frac{1}{z^{2015}} = -i\omega, -i\omega^2$

$\therefore\ z^{2015}-\frac{1}{z^{2015}}=i(\omega^2+\omega)=-i$

8. (d) : $z^{2015}+\frac{1}{z^{2015}}=\pm i(\omega^2-\omega)=\pm\sqrt{3}$

9. (7): The number of ways of getting the sum 9 = coeff. of x^9 in $(x + x^2 + \ldots + x^6)^4$

= coeff. of x^5 in $(1 + x + \ldots + x^5)^4$

$= (1 - x^6)^4(1 - x)^{-4}$

The number of ways of getting the sum atmost 9 is the coeff. of x^5 in $(1 - x)^{-5} = \binom{9}{5} = 126$

Prob. desired $=1-\frac{126}{64}=\frac{65}{72}=\frac{m}{n}$, $\therefore\ n - m = 7$

10. (c): P : $2\sin B = \sin A + \sin C$

$\Rightarrow\ \sin B=\cos\frac{B}{2}\cos\frac{A-C}{2}\le\cos\frac{B}{2}$

$\sin\frac{B}{2}\le\frac{1}{2}\Rightarrow B\le\frac{\pi}{3}$

Q : $r_1 + r_2 = 2R \Rightarrow 4R\cos^2\frac{C}{2}=2R$,

$\Rightarrow\ \cos^2\frac{C}{2}=\frac{1}{2}\ \Rightarrow\ C=\frac{\pi}{2}$

R : $\left[\vec{a}\,\vec{b}\,\vec{c}\right]^2=\begin{vmatrix}1 & \cos\theta & \cos\theta\\ \cos\theta & 1 & \cos\theta\\ \cos\theta & \cos\theta & 1\end{vmatrix}\ge 0$

$\Rightarrow (1 - \cos\theta)^2(1 + 2\cos\theta) \ge 0 \Rightarrow \cos\theta \ge -\frac{1}{2}$,

$\Rightarrow\ \theta\le\frac{2\pi}{3}$. $\therefore$ Maximum value of $\theta=\frac{2\pi}{3}$

S : $\Sigma\tan^{-1}\frac{1}{2r^2}=\Sigma(\tan^{-1}(2r+1)-\tan^{-1}(2r-1))$

$\therefore\ \lim_{n\to\infty}\sum_{r=1}^{n}\ (\tan^{-1}(2r+1)-\tan^{-1}(2r-1))$

$=\lim_{n\to\infty}(\tan^{-1}(2n+1)-\tan^{-1}(1))=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

2015 CBSE-BOARD
CHAPTERWISE PRACTICE PAPER

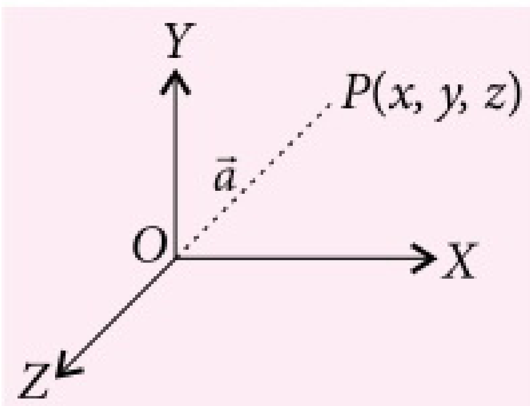
Series-7 | **Vector Algebra and 3D-Geometry**

VECTOR ALGEBRA

- Vector quantities are those quantities which have magnitude as well as direction. A vector is generally represented by a directed line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point.
- The magnitude of vector $\overrightarrow{AB}$ is the length of the line segment AB denoted by $|\overrightarrow{AB}|$.

POSITION VECTOR

- If a point O is fixed in space as origin, then for any point P, the vector $\overrightarrow{OP} = \vec{a}$ is called the position vector (P.V.) of 'P' w.r.t 'O'.
- The magnitude of $\overrightarrow{OP}$ (or $\vec{a}$) is $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$



DIRECTION COSINES

- If α, β, γ are the angles which a vector $\overrightarrow{OP}$ makes with the positive directions of the coordinate axes OX, OY, OZ respectively, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are known as the direction cosines of $\overrightarrow{OP}$ and are generally denoted by the letters, l, m, n respectively.

  The angles α, β, γ are known as the direction angles such that $0 \le \alpha, \beta, \gamma \le \pi$.

DIRECTION RATIOS

- Any three numbers a, b, c proportional to the direction cosines l, m, n respectively of a line, are called direction ratios of the line.

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

NOTE :

(i) The direction cosines of a line are

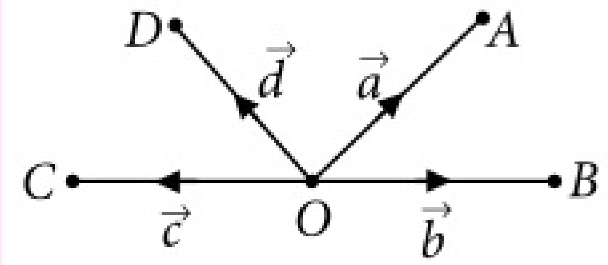
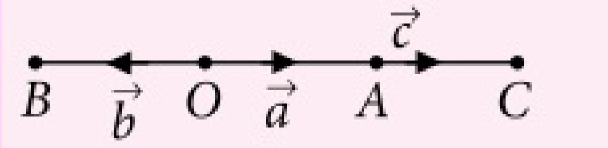
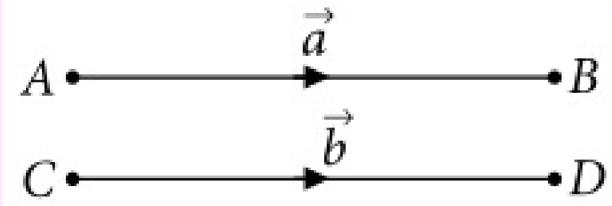
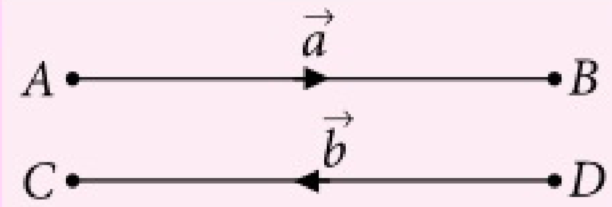
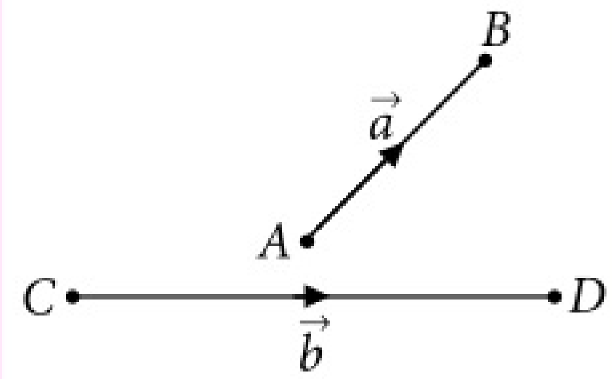
$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}},$$
$$n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}},$$

where a, b, c are direction ratios.

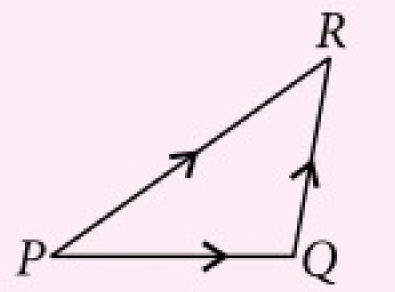
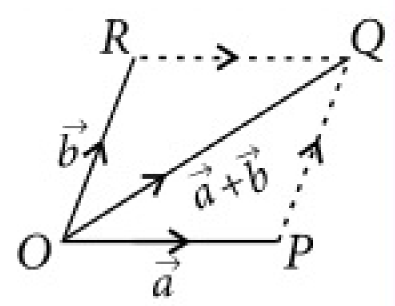
(ii) $l^2 + m^2 + n^2 = 1$

TYPES OF VECTORS

Types of vectors	Definition	Examples
(i) Zero vector or Null vector	A vector whose initial and terminal points are coincident is called the zero vector or the null vector. The magnitude of the zero vector is zero.	$\overrightarrow{AA}$
(ii) Unit vector	A vector of unit magnitude is called a unit vector. Unit vectors are denoted by small letters with a cap on them.	$\hat{a}$ is the unit vector of $\vec{a}$

(iii) Co-initial vectors	Vectors having same initial point are called co-initial vectors. $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are co-initial vectors.	
(iv) Collinear vectors	Vectors which are parallel to the same vector and have either initial or terminal point in common are called collinear vectors. $\vec{a}, \vec{b}, \vec{c}$ are collinear vectors.	
(v) Equal vectors	Two vectors are said to be equal, if they have the same magnitude and same direction. $\vec{a}$ and $\vec{b}$ are equal vectors.	
(vi) Negative of a vector	Two vectors having same magnitude but in opposite direction, are called negative vectors. Ex: $\vec{a} = -\vec{b}$	
(vii) Free vectors	Vectors whose initial point is not specified are called free vectors. $\vec{a}$ and $\vec{b}$ are free vectors	

ADDITION OF VECTORS

Triangle Law		$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$
Parallelogram Law		$\overrightarrow{OP} + \overrightarrow{OR} = \overrightarrow{OQ}$

- **Properties of Addition of Vectors**

For any three vectors, $\vec{a}, \vec{b}, \vec{c}$, we have the following properties :

Commutative law	$\vec{a} + \vec{b} = \vec{b} + \vec{a}$
Associative law	$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
Additive identity	$\vec{a} + \vec{0} = \vec{a} = \vec{0} + \vec{a}$
Additive inverse	$\vec{a} + (-\vec{a}) = \vec{0}$

Multiplication by scalar	$m(\vec{a}) = (\vec{a})m = m\vec{a}$, $m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$, $(m+n)\vec{a} = m\vec{a} + n\vec{a}$, $m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$, where m, n are any two scalars.

COMPONENTS OF A VECTOR

- If $P(x, y, z)$ is a point in space and $\hat{i}, \hat{j}, \hat{k}$ are unit vectors, then

(i) The vector components of $\overrightarrow{OP}$ along X, Y and Z-axes are $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ and x, y and z are called scalar components of $\overrightarrow{OP}$.

(ii) $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$

(iii) $|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

(iv) For any two vectors

$\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

(i) $\vec{a} + \vec{b} = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

(ii) $\vec{a}-\vec{b}=(x_1-x_2)\hat{i}+(y_1-y_2)\hat{j}+(z_1-z_2)\hat{k}$

(iii) $m\vec{a}=(mx_1)\hat{i}+(my_1)\hat{j}+(mz_1)\hat{k}$,

where m is a scalar quantity

(iv) $\vec{a}=\vec{b} \Leftrightarrow x_1=x_2,\ y_1=y_2$ and $z_1=z_2$

VECTOR JOINING TWO POINTS

- Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points, then the vector joining P and Q is given by

$$\overrightarrow{PQ}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}.$$

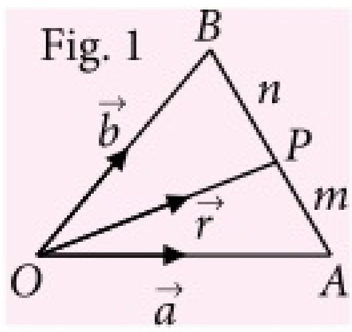
Magnitude of $\overrightarrow{PQ}$ is given by

$$|\overrightarrow{PQ}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

SECTION FORMULA

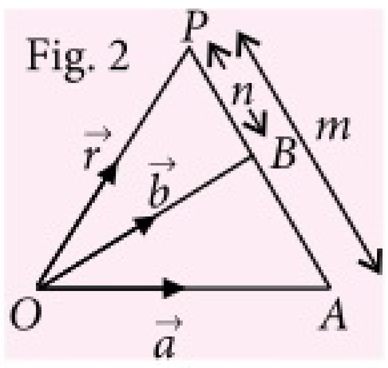
- Let A and B be any two points with position vectors $\vec{a}$ and $\vec{b}$ respectively. P be any point that divides AB in the ratio $m:n$, then

Internal division (Fig. 1) :



$$\overrightarrow{OP}=\vec{r}=\frac{m\vec{b}+n\vec{a}}{m+n}$$

External division (Fig. 2):



$$\overrightarrow{OP}=\vec{r}=\frac{m\vec{b}-n\vec{a}}{m-n}$$

Mid-point formula :

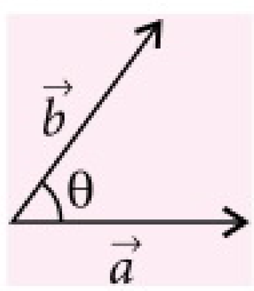
$$\overrightarrow{OP}=\vec{r}=\frac{\vec{a}+\vec{b}}{2}$$

SCALAR (OR DOT) PRODUCT OF TWO VECTORS

- The scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\cdot\vec{b}$, is defined as

$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$,

(where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le \pi$).



If either $\vec{a}=\vec{0}$, $\vec{b}=\vec{0}$, then θ is not defined, and we define $\vec{a}\cdot\vec{b}=0$.

- **Properties of Scalar Product**

(i) $\vec{a}\cdot\vec{b}$ is a real number.

(ii) If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$

(iii) $\vec{a}\cdot\vec{a}=|\vec{a}|^2$

(iv) If $\theta=0$, then $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|$

(v) If $\theta=\pi$, then $\vec{a}\cdot\vec{b}=-|\vec{a}||\vec{b}|$

(vi) For mutually perpendicular unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, we have

$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=\hat{k}\cdot\hat{k}=1$ and $\hat{i}\cdot\hat{j}=\hat{j}\cdot\hat{k}=\hat{k}\cdot\hat{i}=0$

(vii) $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (Commutative law)

(viii) $\vec{a}\cdot(\vec{b}+\vec{c})=\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$ (Distributive law)

(ix) $(\lambda\vec{a})\cdot\vec{b}=\vec{a}\cdot(\lambda\vec{b})$

(x) If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$

(xi) If θ be the angle between $\vec{a}=a_1\hat{i}+b_1\hat{j}+c_1\hat{k}$ and $\vec{b}=a_2\hat{i}+b_2\hat{j}+c_2\hat{k}$, then

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

PROJECTION OF A VECTOR ON A LINE

- Projection of a vector $\vec{a}$ on the other vector $\vec{b}$, is given by

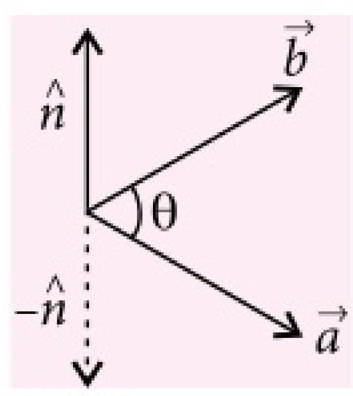
$$\vec{a}\cdot\hat{b},\ \text{or}\ \vec{a}\cdot\left(\frac{\vec{b}}{|\vec{b}|}\right),\ \text{or}\ \frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})$$

- If $\theta=0$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{AB}$ itself and if $\theta=\pi$, then the projection vector of $\overrightarrow{AB}$ will be $\overrightarrow{BA}$.

- If $\theta=\frac{\pi}{2}$ or $\theta=\frac{3\pi}{2}$, then the projection vector of $\overrightarrow{AB}$ will be zero vector.

VECTOR (OR CROSS) PRODUCT OF TWO VECTORS

- The vector product of two non zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}\times\vec{b}$ and is defined as



$$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$$

where, θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \le \theta \le$

π and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, such that $\vec{a}, \vec{b}$ and $\hat{n}$ form a right handed system.

- **Properties of Vector Product**

(i) $\vec{a} \times \vec{b}$ is a vector quantity.

(ii) If $\vec{a}$ and $\vec{b}$ are two non-zero vectors, then $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} \parallel \vec{b}$

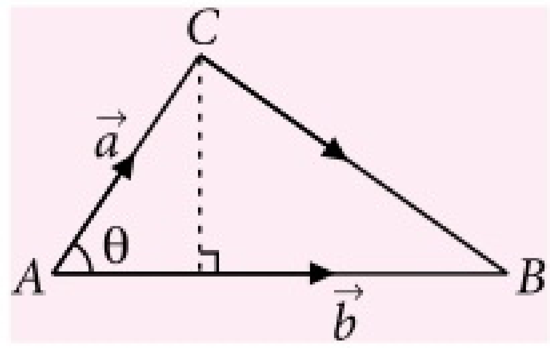
(iii) When $\theta = 0$, then $\vec{a} \times \vec{a} = \vec{0}$

(iv) When $\theta = \pi$, then $\vec{a} \times (-\vec{a}) = \vec{0}$

(v) When $\theta = \dfrac{\pi}{2}$, then $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|$

(vi) For mutually perpendicular unit vectors, $\hat{i}, \hat{j}$ and $\hat{k}$, we have
$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$;
$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

(vii) Area of $\Delta ABC = \dfrac{1}{2}|\vec{a} \times \vec{b}|$,
where $\vec{a}$ and $\vec{b}$ represents the adjacent sides of the triangle.



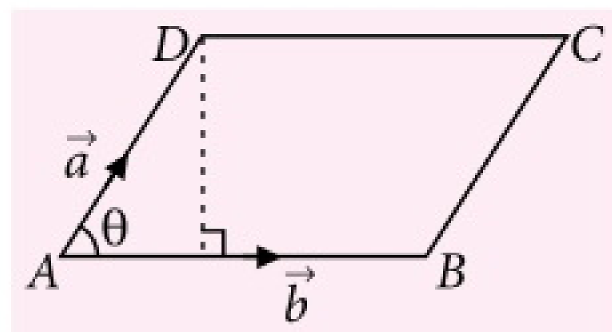
(viii) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ *i.e.*, $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

(ix) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$ (Distributive law)

(x) If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

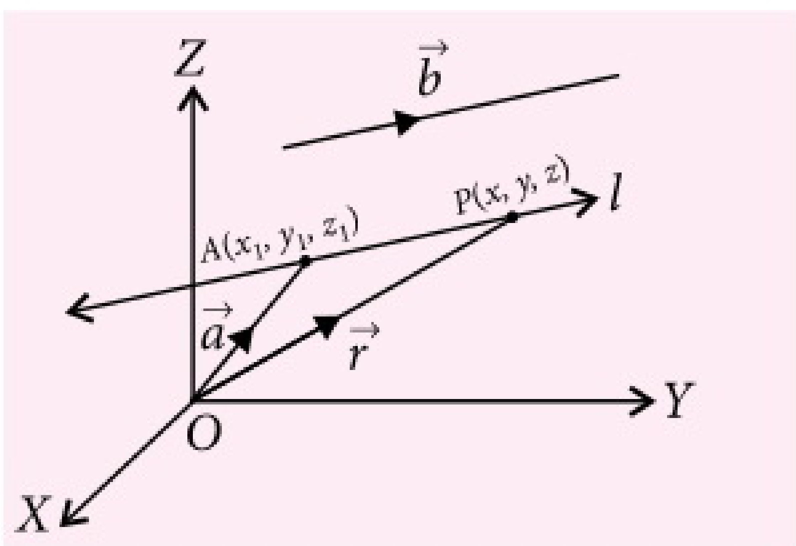
then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

(xi) Area of parallelogram $ABCD = |\vec{a} \times \vec{b}|$
where $\vec{a}$ and $\vec{b}$ represents the adjacent sides of parallelogram



3D-GEOMETRY

EQUATION OF A LINE PASSING THROUGH A GIVEN POINT AND PARALLEL TO A GIVEN VECTOR

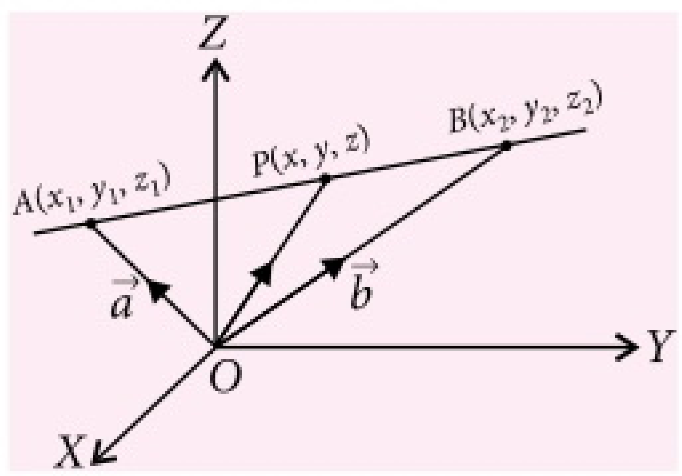
- 

Vector form	$\vec{r} = \vec{a} + \lambda\vec{b}$
Cartesian form	$\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$

If the direction cosines <l, m, n> is given, then equation of line is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}.$$

EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

- 

Vector form	$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}), \lambda \in R$
Cartesian form	$\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$

ANGLE BETWEEN TWO LINES

Vector form	Let the vector equations of two lines be $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$. Then, $\cos\theta = \left|\dfrac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

Cartesian form	Let the cartesian equations of two lines be $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$. Then, $\cos\theta=\left\lvert\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right\rvert$

- **Condition for Parallelism and Perpendicularity of Two Lines**

  Two lines with direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 respectively are

(i) perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) parallel if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

SHORTEST DISTANCE BETWEEN TWO LINES

- **Distance between two skew lines**

Vector form	Let $\vec{r}=a_1+\lambda\vec{b}_1$ and $\vec{r}=a_2+\lambda\vec{b}_2$ represents two skew lines, then the magnitude of shortest distance vector (d) is $d=\left\lvert\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{\lvert\vec{b}_1\times\vec{b}_2\rvert}\right\rvert$
Cartesian form	The shortest distance (d) between the lines, $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ is $d=\left\lvert\frac{\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}\right\rvert$

- **Distance between two parallel lines**

  $\vec{r}=\vec{a}_1+\lambda\vec{b}$ and $\vec{r}=\vec{a}_2+\mu\vec{b}$ is,

$$d=\left\lvert\frac{\vec{b}\times(\vec{a}_2-\vec{a}_1)}{\lvert\vec{b}\rvert}\right\rvert$$

EQUATION OF PLANE

Normal form	$\vec{r}\cdot\hat{n}=d$ (vector form) $lx + my + nz = d$ (cartesian form)
Intercept form	$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

EQUATION OF A PLANE PERPENDICULAR TO A GIVEN VECTOR AND PASSING THROUGH A GIVEN POINT

Vector form	Let $\vec{r}$ be the position vector of any point $P(x, y, z)$ in the plane, then $(\vec{r}-\vec{a})\cdot\vec{n}=0$
Cartesian form	$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

EQUATION OF A PLANE PASSING THROUGH THREE NON-COLLINEAR POINTS :

Vector form	$(\vec{r}-\vec{a})\cdot[(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})]=0$
Cartesian form	$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}=0$

EQUATION OF THE PLANE PASSING THROUGH THE INTERSECTION OF TWO GIVEN PLANES

Vector form	$\vec{r}\cdot(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$
Cartesian form	$(a_1x + b_1y + c_1z - d_1) + \lambda(a_2x + b_2y + c_2z - d_2) = 0$

COPLANARITY OF TWO LINES

Vector form	$(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=0$
Cartesian form	$\begin{vmatrix}x_2-x_1 & y_2-y_1 & z_2-z_1\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}=0$

ANGLE BETWEEN TWO PLANES

Vector form	$\cos\theta = \left\lvert \dfrac{\vec{n}_1 \cdot \vec{n}_2}{\lvert\vec{n}_1\rvert\lvert\vec{n}_2\rvert} \right\rvert$
Cartesian form	$\cos\theta = \left\lvert \dfrac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \right\rvert$

NOTE :

Two planes are said to be

(i) perpendicular if $\cos\theta = a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

DISTANCE OF A POINT FROM A PLANE

Vector form	$\dfrac{\lvert\vec{a}\cdot\vec{n} - d\rvert}{\lvert\vec{n}\rvert}$, where $\vec{n}$ is normal to the plane.
Cartesian form	$\left\lvert \dfrac{ax_1 + by_1 + cz_1 - d}{\sqrt{a^2+b^2+c^2}} \right\rvert$

ANGLE BETWEEN A LINE AND A PLANE

$$\theta = \sin^{-1}\left\lvert \frac{\vec{b}\cdot\vec{n}}{\lvert\vec{b}\rvert\lvert\vec{n}\rvert} \right\rvert$$

VERY SHORT ANSWER TYPE (1 MARK)

1. If $\overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{QO} + \overrightarrow{OR}$, show that the points P, Q, R are collinear.

2. The position vectors of points A, B, C, D are $\vec{a}, \vec{b}, 2\vec{a} + 3\vec{b}$ and $\vec{a} - 2\vec{b}$ respectively. Find $\overrightarrow{DB}$ and $\overrightarrow{AC}$.

3. Find the value of p for which the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\vec{b} = \hat{i} + p\hat{j} + 3\hat{k}$ are perpendicular.

4. Find the equation of the plane with intercepts 2, 3 and 4 on the X, Y and Z-axes, respectively.

5. Find the angle between the two vectors $\vec{a}$ and $\vec{b}$ with magnitudes 1 and 2, respectively and $\vec{a}\cdot\vec{b} = 1$.

SHORT ANSWER TYPE (4 MARKS)

6. If $\vec{a} = 3\hat{i} - 2\hat{j} - 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$, calculate $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$.

7. Find the vector and cartesian equation of the plane that passes through the point $(1, 4, 6)$ and the normal vector to the plane is $\hat{i} - 2\hat{j} + \hat{k}$.

8. A vector $\vec{r}$ has length 21 units and direction ratios 2, –3, 6. Find the direction cosines and vector components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with x-axis.

9. Show that the area of a parallelogram having diagonals $3\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} - 3\hat{j} + 4\hat{k}$ is $5\sqrt{3}$.

10. Show that the lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-4}{5} = \dfrac{y-1}{2} = z$ intersect. Also find the point of intersection of these lines.

LONG ANSWER TYPE (6 MARKS)

11. Find the vector equation of the line passing through the point (2, 3, 2) and parallel to the line $\vec{r} = -2\hat{i} + 3\hat{j} + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance between the lines.

12. Find the image of the line $\dfrac{x-1}{3} = \dfrac{y-3}{1} = \dfrac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$.

13. Find the shortest distance between the lines
$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$$
and $\vec{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 8\hat{k})$

14. Prove that the lines $\dfrac{x-2}{1} = \dfrac{y-4}{4} = \dfrac{z-6}{7}$ and $\dfrac{x+1}{3} = \dfrac{y+3}{5} = \dfrac{z+5}{7}$ are coplanar. Also, find the equation of plane containing these lines.

15. If $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$, then find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}\cdot\vec{d}=15$.

SOLUTIONS

1. We have, $\overrightarrow{PO}+\overrightarrow{OQ}=\overrightarrow{QO}+\overrightarrow{OR}$

$\Rightarrow \overrightarrow{PQ}=\overrightarrow{QR}$ [By triangle law]

Thus, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are either parallel or collinear. But, Q is a point common to them.

So, $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are collinear.

Hence, points P, Q, R are collinear.

2. We have,

$\overrightarrow{DB}$ = Position vector of B – Position vector of D

$\Rightarrow \overrightarrow{DB}=\vec{b}-(\vec{a}-2\vec{b})=3\vec{b}-\vec{a}$

Similarly, $\overrightarrow{AC}=(2\vec{a}+3\vec{b})-\vec{a}=\vec{a}+3\vec{b}$

3. If vectors $\vec{a}$ and $\vec{b}$ are perpendicular, then

$\vec{a}\cdot\vec{b}=0$

$\Rightarrow (3\hat{i}+2\hat{j}+9\hat{k})\cdot(\hat{i}+p\hat{j}+3\hat{k})=0$

$\Rightarrow 3+2p+27=0 \Rightarrow p=-15$

4. We know that the equation of plane in intercept form is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Here, $a = 2$, $b = 3$ and $c = 4$

$\therefore \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1 \Rightarrow 6x+4y+3z=12$

5. Given, $|\vec{a}|=1$, $|\vec{b}|=2$ and $\vec{a}\cdot\vec{b}=1$

$$\therefore \theta=\cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)=\cos^{-1}\left(\frac{1}{2}\right)=60^\circ$$

6. Given, $\vec{a}=3\hat{i}-2\hat{j}-2\hat{k}$ and $\vec{b}=2\hat{i}+3\hat{j}+\hat{k}$

$\therefore \vec{a}+\vec{b}=5\hat{i}+\hat{j}-\hat{k}$

and $\vec{a}-\vec{b}=\hat{i}-5\hat{j}-3\hat{k}$

$$\therefore (\vec{a}+\vec{b})\times(\vec{a}-\vec{b})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 1 & -1\\ 1 & -5 & -3\end{vmatrix}$$

$=\hat{i}(-3-5)-\hat{j}(-15+1)+\hat{k}(-25-1)$

$=-8\hat{i}+14\hat{j}-26\hat{k}$

7. Let $\vec{a}$ be the position vector of the point (1, 4, 6) and $\vec{n}$ be the normal vector perpendicular to the plane.

$\therefore \vec{a}=\hat{i}+4\hat{j}+6\hat{k}$ and $\vec{n}=\hat{i}-2\hat{j}+\hat{k}$

$\therefore$ The vector equation of plane is $(\vec{r}-\vec{a})\cdot\vec{n}=0$

$\Rightarrow [\vec{r}-(\hat{i}+4\hat{j}+6\hat{k})]\cdot(\hat{i}-2\hat{j}+\hat{k})=0$

$\Rightarrow \vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k})-(\hat{i}+4\hat{j}+6\hat{k})\cdot(\hat{i}-2\hat{j}+\hat{k})=0$

$\Rightarrow \vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k})-(1-8+6)=0$

$\Rightarrow \vec{r}\cdot(\hat{i}-2\hat{j}+\hat{k})=-1$

Also, the cartesian equation of plane is

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\hat{i}-2\hat{j}+\hat{k})=-1$

$\Rightarrow x-2y+z=-1 \Rightarrow x-2y+z+1=0$

8. Recall that if the direction ratios of a vector are a, b, c, then its direction cosines are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

Therefore, direction cosines of $\vec{r}$ are

$$\pm\frac{2}{\sqrt{2^2+(-3)^2+6^2}}, \pm\frac{-3}{\sqrt{2^2+(-3)^2+6^2}}, \pm\frac{6}{\sqrt{2^2+(-3)^2+6^2}}$$

Since, $\vec{r}$ makes an acute angle with x-axis.

Therefore, $\cos\alpha > 0$, *i.e.*, $l > 0$.

So, direction cosines of $\vec{r}$ are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$

$$\therefore \vec{r}=21\left(\frac{2}{7}\hat{i}-\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k}\right) \Rightarrow \vec{r}=6\hat{i}-9\hat{j}+18\hat{k}$$

So, vector components of $\vec{r}$ along X, Y and Z axes are $6\hat{i}$, $-9\hat{j}$ and $18\hat{k}$ respectively.

9. Let $\vec{a}=3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{b}=\hat{i}-3\hat{j}+4\hat{k}$, then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & -2\\ 1 & -3 & 4\end{vmatrix}$$

$=(4-6)\hat{i}-(12+2)\hat{j}+(-9-1)\hat{k}=-2\hat{i}-14\hat{j}-10\hat{k}$

$\Rightarrow |\vec{a}\times\vec{b}|=\sqrt{(-2)^2+(-14)^2+(-10)^2}=\sqrt{300}$

$\therefore$ Area of parallelogram $=\frac{1}{2}|\vec{a}\times\vec{b}|=\frac{1}{2}\sqrt{300}$

$=5\sqrt{3}$ sq. units

10. The given lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad \text{... (i)}$$

and $$\frac{x-4}{5}=\frac{y-1}{2}=\frac{z-0}{1} \quad \text{... (ii)}$$

Any point on the line (i) is $(1 + 2t, 2 + 3t, 3 + 4t)$

It will lie on the line (ii) iff

$\frac{1+2t-4}{5}=\frac{2+3t-1}{2}=\frac{3+4t-0}{1}$ for a unique t,

i.e., iff $15t + 5 = 4t - 6$ and $8t + 6 = 3t + 1$ for a unique t.

i.e., iff $t = -1$ and $t = -1$, which is true.

Therefore, the given lines intersect and the point of intersection is

$(1 + 2(-1), 2 + 3(-1), 3 + 4(-1))$, *i.e.*, $(-1, -1, -1)$.

11. The given line is

$$\vec{r}=-2\hat{i}+3\hat{j}+\lambda(2\hat{i}-3\hat{j}+6\hat{k}) \quad \text{... (i)}$$

It passes through the point A with position vector $\vec{a}=-2\hat{i}+3\hat{j}$, and it is parallel to the vector $\vec{v}=2\hat{i}-3\hat{j}+6\hat{k}$.

The equation of the line passing through the point B (2, 3, 2) with position vector

$\vec{b}=2\hat{i}+3\hat{j}+2\hat{k}$ and parallel to the line (i) is

$$\vec{r}=2\hat{i}+3\hat{j}+2\hat{k}+\mu(2\hat{i}-3\hat{j}+6\hat{k}) \quad \text{... (ii)}$$

Also, $\vec{b}-\vec{a}=4\hat{i}+2\hat{k}$,

$$|\vec{v}|=\sqrt{2^2+(-3)^2+6^2}=7$$

and $$\vec{v}\times(\vec{b}-\vec{a})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 6\\ 4 & 0 & 2\end{vmatrix}$$

$$=-6\hat{i}+20\hat{j}+12\hat{k}$$

$$\therefore \quad |\vec{v}\times(\vec{b}-\vec{a})|=\sqrt{(-6)^2+(20)^2+(12)^2}=\sqrt{580}$$

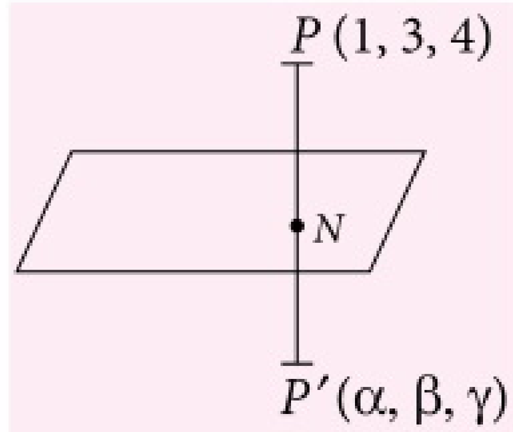
$\therefore$ The distance between the lines

$$=\left|\frac{\vec{v}\times(\vec{b}-\vec{a})}{|\vec{v}|}\right|=\frac{\sqrt{580}}{7} \text{ units}$$

12. The given line is $\frac{x-1}{3}=\frac{y-3}{1}=\frac{z-4}{-5}$ ... (i)

and the given plane is $2x - y + z + 3 = 0$ ... (ii)

Since $3 \times 2 + 1 \times (-1) + (-5) \times 1 = 0$ and the point $P(1, 3, 4)$ of line (i) does not lie in plane (ii), therefore, line (i) is parallel to the plane (ii).



Let $P'(\alpha, \beta, \gamma)$ be the image of the point $P(1, 3, 4)$ in the plane (ii), then the image of line (i) in the plane (ii) is a line through P' and parallel to line (i).

Let N be the foot of perpendicular from P onto the plane (ii), then $[PN] \perp$ (ii). Direction ratio of line PN is $<2, -1, 1>$ and hence, equation of line PN is

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1} \quad \text{... (iii)}$$

Since N lies on (iii), we can take N as

$(2t + 1, -t + 3, t + 4)$

But, N lies in the plane (ii), therefore

$$2(2t + 1) - (-t + 3) + t + 4 + 3 = 0$$

$\Rightarrow \quad 6t + 6 = 0 \Rightarrow t = -1$

$\therefore \quad N \equiv (-1, 4, 3)$.

Also, N is the mid-point of PP'.

$$\therefore \quad (-1, 4, 3)=\left(\frac{1+\alpha}{2}, \frac{3+\beta}{2}, \frac{4+\gamma}{2}\right)$$

$\Rightarrow \quad \alpha = -3, \beta = 5, \gamma = 2$

$\therefore \quad P'$ is $(-3, 5, 2)$ and hence, equation of the image of the line (i) in the plane (ii) are

$$\frac{x-(-3)}{3}=\frac{y-5}{1}=\frac{z-2}{-5} \text{ or } \frac{x+3}{3}=\frac{y-5}{1}=\frac{z-2}{-5}$$

13. The given lines are

$$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}+4\hat{k}) \quad \text{... (i)}$$

and $$\vec{r}=(2\hat{i}+4\hat{j}+5\hat{k})+2\mu(2\hat{i}+3\hat{j}+4\hat{k}) \quad \text{... (ii)}$$

Equation (ii) can be re-written as

$$\vec{r}=(2\hat{i}+4\hat{j}+5\hat{k})+\mu'(2\hat{i}+3\hat{j}+4\hat{k}) \quad \text{... (iii)}$$

where $\mu' = 2\mu$

These two lines passes through the points having position vectors $\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{a}_2=2\hat{i}+4\hat{j}+5\hat{k}$ respectively and both are parallel to the vector $\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$.

Now,

$$\vec{a}_2 - \vec{a}_1 = \hat{i} + 2\hat{j} + 2\hat{k}$$

$$\Rightarrow (\vec{a}_2 - \vec{a}_1) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 3 & 4 \end{vmatrix} = 2\hat{i} - 0\hat{j} - \hat{k}$$

$$\Rightarrow |(\vec{a}_2 - \vec{a}_1) \times \vec{b}| = \sqrt{4+0+1} = \sqrt{5}$$

and $|\vec{b}| = \sqrt{4+9+16} = \sqrt{29}$

Now, shortest distance $= \dfrac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|} = \sqrt{\dfrac{5}{29}}.$

14. Given lines are

$$\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7} \text{ and } \frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$$

Here, points and direction ratios of given lines are

$x_1 = 2, y_1 = 4, z_1 = 6$ and $a_1 = 1, b_1 = 4, c_1 = 7$; $x_2 = -1, y_2 = -3, z_2 = -5$ and $a_2 = 3, b_2 = 5, c_2 = 7$ respectively.

Now, if above lines are coplanar, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\therefore \begin{vmatrix} -1-2 & -3-4 & -5-6 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix} = \begin{vmatrix} -3 & -7 & -11 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix}$$

$= -3(28 - 35) + 7(7 - 21) - 11(5 - 21)$
$= 21 - 98 + 77 = 0$

Hence, the given lines are coplanar.

Now, equation of the plane containing these lines is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-2 & y-4 & z-6 \\ 1 & 4 & 7 \\ 3 & 5 & 7 \end{vmatrix} = 0$$

Expanding along R_1, we get

$$(x-2)(28-35) - (y-4)(7-21) + (z-6)(5-12) = 0$$

$$\Rightarrow -7(x-2) + 14(y-4) - 7(z-6) = 0$$

$$\Rightarrow -7x + 14 + 14y - 56 - 7z + 42 = 0$$

$$\Rightarrow -7x + 14y - 7z = 0 \Rightarrow x - 2y + z = 0$$

15. Here, $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= \hat{i}(28+4) - \hat{j}(7-6) + \hat{k}(-2-12)$$

$$= 32\hat{i} - \hat{j} - 14\hat{k}$$

Since, $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

So, vector $\vec{d}$ is parallel to $\vec{a} \times \vec{b}$.

$$\therefore \vec{d} = \lambda(\vec{a} \times \vec{b}) = \lambda(32\hat{i} - \hat{j} - 14\hat{k})$$

$$= 32\lambda\hat{i} - \lambda\hat{j} - 14\lambda\hat{k}$$

Now, $\vec{c} \cdot \vec{d} = 15$

$$\therefore (2\hat{i} - \hat{j} + 4\hat{k}) \cdot (32\lambda\hat{i} - \lambda\hat{j} - 14\lambda\hat{k}) = 15$$

$$\Rightarrow 64\lambda + \lambda - 56\lambda = 15$$

$$\Rightarrow 9\lambda = 15 \Rightarrow \lambda = \frac{15}{9} = \frac{5}{3}$$

$\therefore$ Required vector is

$$\vec{d} = \frac{160}{3}\hat{i} - \frac{5}{3}\hat{j} - \frac{70}{3}\hat{k}$$

concept BOOSTERS

Class XI

Trigonometry

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This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

BASIC TRIGONOMETRIC IDENTITIES

- $\text{cosec}^2\theta - \cot^2\theta = 1$; $|\text{cosec}\,\theta| \ge 1 \;\; \forall \;\; \theta \in R$
- $\sin^2\theta + \cos^2\theta = 1$; $-1 \le \sin\theta \le 1$; $-1 \le \cos\theta \le 1 \;\; \forall \;\; \theta \in R$
- $\sec^2\theta - \tan^2\theta = 1$; $|\sec\theta| \ge 1 \;\forall\; \theta \in R$

RESULTS

- $\sin\dfrac{\pi}{8} = \dfrac{\sqrt{2-\sqrt{2}}}{2}$; $\cos\dfrac{\pi}{8} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$

  $\tan\dfrac{\pi}{8} = \sqrt{2}-1$; $\tan\dfrac{3\pi}{8} = \sqrt{2}+1$
- $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$, where $n \in I$
- $\sin\dfrac{\pi}{10}$ or $\sin 18° = \dfrac{\sqrt{5}-1}{4}$ and

  $\cos 36°$ or $\cos\dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4}$
- $\sin\dfrac{(2n+1)\pi}{2} = (-1)^n$ and $\cos\dfrac{(2n+1)\pi}{2} = 0$, where $n \in I$
- $\sin 15°$ or $\sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75°$ or $\cos\dfrac{5\pi}{12}$;

  $\cos 15°$ or $\cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75°$ or $\sin\dfrac{5\pi}{12}$;

  $\tan 15° = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3} = \cot 75°$;

  $\tan 75° = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3} = \cot 15°$

TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES

If θ is any angle, then $-\theta$, $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$ etc. are called allied angles.

only sin and cosec +ve	All +ve
only tan and cot +ve	only cos and sec +ve

- $\sin(90°+\theta) = \cos\theta$; $\cos(90°+\theta) = -\sin\theta$
- $\sin(270°-\theta) = -\cos\theta$; $\cos(270°-\theta) = -\sin\theta$
- $\sin(270°+\theta) = -\cos\theta$; $\cos(270°+\theta) = \sin\theta$
- $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$
- $\sin(90°-\theta) = \cos\theta$; $\cos(90°-\theta) = \sin\theta$
- $\sin(180°-\theta) = \sin\theta$; $\cos(180°-\theta) = -\cos\theta$
- $\sin(180°+\theta) = -\sin\theta$; $\cos(180°+\theta) = -\cos\theta$

TRIGONOMETRIC FUNCTIONS OF SUM OR DIFFERENCE OF TWO ANGLES

- $\cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A = \cos(A+B)\cdot\cos(A-B)$
- $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
- $\cot(A \pm B) = \dfrac{\cot A \cot B \mp 1}{\cot B \pm \cot A}$

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- $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
- $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
- $\sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$
  $= \sin(A + B)\cdot\sin(A - B)$

FACTORISATION OF THE SUM OR DIFFERENCE OF TWO SINES OR COSINES

- $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$
- $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$
- $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

TRANSFORMATION OF PRODUCTS INTO SUM OR DIFFERENCE OF SINES & COSINES

- $2\sin A \cos B = \sin(A + B) + \sin(A - B)$
- $2\cos A \sin B = \sin(A + B) - \sin(A - B)$
- $2\cos A \cos B = \cos(A + B) + \cos(A - B)$
- $2\sin A \sin B = \cos(A - B) - \cos(A + B)$

MULTIPLE ANGLES AND HALF ANGLES

- $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$; $\tan A = \dfrac{2\tan(A/2)}{1 - \tan^2(A/2)}$
- $\sin 2A = 2\sin A \cos A$; $\sin A = 2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)$
- $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1$
  $= 1 - 2\sin^2 A$;
  $\cos A = \cos^2\left(\frac{A}{2}\right) - \sin^2\left(\frac{A}{2}\right) = 2\cos^2\left(\frac{A}{2}\right) - 1$
  $= 1 - 2\sin^2\left(\frac{A}{2}\right)$
  $2\cos^2 A = 1 + \cos 2A$, $2\sin^2 A = 1 - \cos 2A$;
  $\tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$
  $2\cos^2\left(\frac{A}{2}\right) = 1 + \cos A$, $2\sin^2\left(\frac{A}{2}\right) = 1 - \cos A$
- $\sin 2A = \dfrac{2\tan A}{1 + \tan^2 A}$, $\cos 2A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$
- $\sin 3A = 3\sin A - 4\sin^3 A$
- $\cos 3A = 4\cos^3 A - 3\cos A$
- $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

THREE ANGLES

- $\tan(A + B + C)$
  $= \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$
  (i) If $A + B + C = \pi$, then
  $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
  (ii) If $A + B + C = \pi/2$, then
  $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$
- If $A + B + C = \pi$, then
  (i) $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
  (ii) $\sin A + \sin B + \sin C = 4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right)$

MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC FUNCTIONS

- If a quadratic in $\sin\theta$ or $\cos\theta$ is given, then the maximum or minimum values can be interpreted by making a perfect square.
- If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ and $\alpha + \beta = \sigma$(constant), then the minimum values of the expression $\sec\alpha + \sec\beta$, $\tan\alpha + \tan\beta$, $\mathrm{cosec}\,\alpha + \mathrm{cosec}\,\beta$ occurs when $\alpha = \beta = \sigma/2$.
- Min. value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$, where $\theta \in R$
- Max. and min. value of $a\cos\theta + b\sin\theta$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$
- If $f(\theta) = a\cos(\alpha + \theta) + b\cos(\beta + \theta)$ where a, b, α and β are known quantities, then
  $$-\sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)} \le f(\theta) \le \sqrt{a^2 + b^2 + 2ab\cos(\alpha - \beta)}$$
- If $\alpha, \beta \in (0, \pi/2)$ and $\alpha + \beta = \sigma$ (constant), then the maximum values of the expression $\cos\alpha \cos\beta$, $\cos\alpha + \cos\beta$, $\sin\alpha + \sin\beta$ and $\sin\alpha \sin\beta$ occurs when $\alpha = \beta = \sigma/2$.
- If A, B, C are the angles of a triangle, then maximum value of $\sin A + \sin B + \sin C$ and $\sin A \sin B \sin C$ occurs when $A = B = C = 60°$

- Sum of sines or cosines of n angles,
$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots\ldots$
$$\ldots + \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)$$
$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots$
$$\ldots. + \cos(\alpha + \overline{n-1}\,\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)$$

SOLUTION OF TRIGONOMETRIC EQUATION

- $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$,where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$
- $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$, where $\alpha \in [0, \pi]$, $n \in I$.
- $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$, where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $n \in I$
- $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$
- $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$
- $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$
  [Note : α is called the principal angle]

TYPES OF TRIGONOMETRIC EQUATIONS

- Solving equations by transforming a product of trigonometric functions into a sum.
  Consider the equations,
  $\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$;
  $8\cos x \cos 2x \cos 4x = \dfrac{\sin 6x}{\sin x}$;
  $\sin 3\theta = 4\sin\theta \sin 2\theta \sin 4\theta$
- Solving equations by introducing an auxiliary argument. Consider the equations,
  $\sin x + \cos x = \sqrt{2}$; $\sqrt{3}\cos x + \sin x = 2$;
  $\sec x - 1 = (\sqrt{2} - 1)\tan x$
- Solving equations by transforming a sum of trigonometric functions into a product.
  Consider the equations,
  $\cos 3x + \sin 2x - \sin 4x = 0$;
  $\sin^2 x + \sin^2 2x + \sin^2 3x + \sin^2 4x = 2$;
  $\sin x + \sin 5x = \sin 2x + \sin 4x$
- Solution of equations by factorising . Consider the equations,
  $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$;
  $\cot x - \cos x = 1 - \cot x \cos x$
- Solution of equations reducible to quadratic equations. Consider the equations,
  $3\cos^2 x - 10\cos x + 3 = 0$ and
  $2\sin^2 x + \sqrt{3}\sin x + 1 = 0$
- Solving equations by a change of variable :
  (i) Equations of the form of $a\sin x + b\cos x + d = 0$, where a, b and d are real numbers and $a, b \neq 0$ can be solved by changing $\sin x$ and $\cos x$ into their corresponding tangent of half the angle. Consider the equation $3\cos x + 4\sin x = 5$.
  (ii) Many equations can be solved by introducing a new variable. *e.g.* the equation $\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to $2(y+1)\left(y - \frac{1}{2}\right) = 0$ by substituting, $\sin 2x \cdot \cos 2x = y$.
- Solving equations with the use of the boundness of the functions $\sin x$ and $\cos x$ or by making two perfect squares. Consider the equations,
  $$\sin x\left(\cos\frac{x}{4} - 2\sin x\right) + \left(1 + \sin\frac{x}{4} - 2\cos x\right)\cos x = 0;$$
  $$\sin^2 x + 2\tan^2 x + \frac{4}{\sqrt{3}}\tan x - \sin x + \frac{11}{12} = 0$$

TRIGONOMETRIC INEQUALITIES

There is no general rule to solve a trigonometric inequation and the same rules of algebra are valid except the domain and range of trigonometric functions should be kept in mind.
Consider the examples :
$$\log_2\left(\sin\frac{x}{2}\right) < -1;\quad \sin x\left(\cos x + \frac{1}{2}\right) \leq 0;$$
$$\sqrt{5 - 2\sin 2x} \geq 6\sin x - 1$$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. If the angles A, B and C of a triangle are in A.P. and if a, b and c denote the length of sides opposite to A, B and C respectively, then the value of $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A =$
(a) 1/2 (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\sqrt{3}$

2. AB is a line segment of length 24 cm and C is its middle point. On AB, AC and CB semi-circles are described. The radius of circle which touches all these semi-circles is

(a) 3 cm (b) 6 cm (c) 4 cm (d) 8 cm

3. In a ΔABC, $\Delta = 6$ sq. units, $abc = 60$ cu. units, $r = 1$ unit. Then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=$ __ units (nearly)

(a) 0.5 (b) 0.6

(c) 0.4 (d) 0.8

4. If $x=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$ and $y=\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}$, then $x^2 + y^2$ is

(a) 1 (b) 2 (c) 3 (d) 4

5. In a triangle ABC, the medians from B and C are perpendicular. The value of $\cot B + \cot C$ cannot be equal to

(a) 1/3 (b) 2/3

(c) 4/3 (d) none of these

6. In a scalene ΔABC, D is a point on the side AB such that $CD^2 = AD \cdot DB$, if $\sin A\cdot\sin B=\sin^2\left(\frac{C}{2}\right)$, then CD

(a) is a median through C

(b) is an internal bisector of $\angle C$

(c) is an altitude through C

(d) divides AB in the ratio 1 : 2

7. A sector subtends an angle 2α at the centre, then the greatest area of the rectangle inscribed in the sector is (R is radius of the circle)

(a) $R^2\tan\frac{\alpha}{2}$ (b) $\frac{R^2}{2}\tan\frac{\alpha}{2}$

(c) $R^2\tan\alpha$ (d) $\frac{R^2}{2}\tan\alpha$

8. The minimum value of $\sin(\cos x) + \cos(\sin x)$, for $x\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$ is

(a) $\cos(\cos 1)$ (b) $1 + \cos 1$

(c) $\cos 1$ (d) $1 + \sin 1$

9. Let $f(x) = \sin^{23}x - \cos^{22}x$ and $g(x)=1+\frac{1}{2}\tan^{-1}|x|$, then the number of values of 'x' in the interval $[-10\pi, 20\pi]$ satisfying the equation $f(x) = \mathrm{sgn}(g(x))$ is___ (where 'sgn' is signum function)

(a) 6 (b) 10 (c) 15 (d) 20

10. Which of the following quantities is negative?

(a) $\cos(\tan^{-1}(\tan 4))$ (b) $\sin(\cot^{-1}(\cot 4))$

(c) $\tan(\cos^{-1}(\cos 5))$ (d) $\cot(\sin^{-1}(\sin 4))$

11. If $P_n = \cos^n\theta + \sin^n\theta$, then $2P_6 - 3P_4$ is

(a) 0 (b) -1 (c) 1 (d) 5

12. In a triangle ABC, if $A - B = 120°$, $R = 8r$, then $\sin\frac{C}{2}=$

(a) 1/4 (b) 1/2 (c) 1/3 (d) 2/3

13. The real numbers α and β lie in the interval $\left(\frac{\pi}{2},\pi\right)$. If $2|\sin 2\alpha| = |\tan\beta + \cot\beta|$, what is the value of $\alpha + \beta$?

(a) $\frac{3\pi}{4}$ (b) π (c) $\frac{3\pi}{2}$ (d) $\frac{5\pi}{4}$

14. The number of distinct real roots of the equation $\begin{vmatrix}\sin x & \cos x & \cos x\\ \cos x & \sin x & \cos x\\ \cos x & \cos x & \sin x\end{vmatrix}=0$ in $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$ is

(a) 0 (b) 1 (c) 2 (d) 3

15. Consider the system of linear equations in x, y and z, $(\sin 3\theta)x - y + z = 0$, $(\cos 2\theta)x + 4y + 3z = 0$, $2x + 7y + 7z = 0$, then which of the following can be the values of θ for which the system has a non-trivial solution?

(a) $n\pi+(-1)^n\frac{\pi}{6}, \forall n\in Z$

(b) $n\pi+(-1)^n\frac{\pi}{3}, \forall n\in Z$

(c) $n\pi+(-1)^n\frac{\pi}{9}, \forall n\in Z$

(d) $2n\pi+\frac{\pi}{10}, \forall n\in Z$

16. Number of solutions of $3^{\sin 2x+2\cos^2 x}+3^{1+2\sin x(\sin x-\cos x)}=28$ in $[0, 2\pi]$ is

(a) 3 (b) 4 (c) 5 (d) 6

17. In a ΔABC, A and B are two solutions of $k\sec^2 x = 2\tan x$ $(0 < k < 1, A \neq B)$, then $\sin C =$

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{3}}$

18. $x \sin a + y \sin 2a + z \sin 3a = \sin 4a$
$x \sin b + y \sin 2b + z \sin 3b = \sin 4b$
$x \sin c + y \sin 2c + z \sin 3c = \sin 4c.$
Then the roots of the equation

$t^3 - \frac{z}{2}t^2 - \frac{y+2}{4}t + \frac{z-x}{8} = 0$ $(a, b, c \neq n\pi)$ are

(a) $\sin a, \sin b, \sin c$ (b) $\cos a, \cos b, \cos c$
(c) $\sin 2a, \sin 2b, \sin 2c$ (d) $\cos 2a, \cos 2b, \cos 2c$

19. Let $f(\theta) = \sin\theta(\sin\theta + \sin 3\theta)$, then $f(\theta)$
(a) ≥ 0 only when $\theta \geq 0$
(b) ≤ 0 for all real θ
(c) ≥ 0 for all real θ
(d) ≤ 0 only when $\theta \leq 0$

20. The least positive value of x satisfying $\frac{\sin^2 2x + 4\sin^4 x - 4\sin^2 x \cos^2 x}{4 - \sin^2 2x - 4\sin^2 x} = \frac{1}{9}$ is

(a) $\pi/3$ (b) $\pi/6$ (c) $2\pi/3$ (d) $5\pi/6$

21. The number of ordered pairs (x, y) satisfying $|x| + |y| = 2$ and $\sin\left(\frac{\pi x^2}{3}\right) = 1$ is/are

(a) 1 (b) 2 (c) 3 (d) 4

22. Number of solutions of

$\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] - \tan^2\left[x + \frac{\pi}{4}\tan^2 x\right] = 1,$

$x \in [-2\pi, 2\pi]$ is

(a) 1 (b) 2 (c) 4 (d) 8

23. If the equation $x^2 + 5 + 4\cos(\alpha x + \beta) = 2x$ has at least one solution where $\alpha, \beta \in [2, 5]$, then the value of $(\alpha + \beta)$ is equal to

(a) π (b) 2π (c) 3π (d) 5π

24. The number of distinct solutions of $\sin 5\theta \cdot \cos 3\theta = \sin 9\theta \cdot \cos 7\theta$ in $[0, \pi/2]$ is

(a) 4 (b) 5 (c) 8 (d) 9

25. The maximum value of the expression $\left|\sqrt{\sin^2 x + 2a^2} - \sqrt{2a^2 - 1 - \cos^2 x}\right|$, where a and x are real numbers, is

(a) 1 (b) 2 (c) $\sqrt{2}$ (d) $\sqrt{3}$

SECTION-II

More than One Correct Answer Type

26. If $p = \sin(A - B)\cdot\sin(C - D)$
$q = \sin(B - C)\cdot\sin(A - D)$
$r = \sin(C - A)\cdot\sin(B - D)$, then

(a) $p + q - r = 0$ (b) $p + q + r = 0$
(c) $p - q + r = 0$ (d) $p^3 + q^3 + r^3 = 3pqr$

27. If $(a + 2)\sin A + (2a - 1)\cos A = (2a + 1)$, then one of the possible value of $\tan A$ is

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{2a}{a^2+1}$ (d) $\frac{2a}{a^2-1}$

28. If $\sin x + \sin y = \frac{1}{4}$, $\cos x + \cos y = \frac{1}{3}$, then

(a) $\tan\left(\frac{x+y}{2}\right) = \frac{3}{4}$ (b) $\tan(x+y) = \frac{24}{7}$

(c) $\cos(x+y) = \frac{7}{25}$ (d) $\sin(x+y) = \frac{5}{6}$

29. Which of the following is true?

(a) If $b\sin A = a$, $\angle A < \frac{\pi}{2} \Rightarrow \Delta ABC$ is possible.

(b) If $b\sin A > a$, $\angle A > \frac{\pi}{2} \Rightarrow \Delta ABC$ is possible.

(c) If $b\sin A < a$, $\angle A < \frac{\pi}{2}$, $b > a \Rightarrow \Delta ABC$ is possible.

(d) If $b\sin A > a$, $\angle A < \frac{\pi}{2} \Rightarrow \Delta ABC$ is possible.

30. In ΔABC, c_1, c_2 are two values of side 'c' in the ambiguous case for given a, b, A, then

(a) $c_1 + c_2 = 2b\cos A$ (b) $c_1 c_2 = b^2 + a^2$

(c) $|c_1 - c_2| = 2\sqrt{a^2 - b^2\sin^2 A}$

(d) $|c_1 - c_2| = 2\sqrt{a^2 - b^2\cos^2 A}$

31. If in ΔABC, $B = 60°$, then
(a) $(a - b)^2 = c^2 - ab$
(b) $(c - a)^2 = b^2 - ac$
(c) $(b - c)^2 = a^2 - bc$
(d) $a^2 + b^2 + c^2 = 2b^2 + ac$

32. If $\left(\cos^2 x + \frac{1}{\cos^2 x}\right)(1 + \tan^2 2y)(3 + \sin 3z) = 4,$

then

(a) x may be a multiple of π
(b) x cannot be an even multiple of π

(c) z can be a multiple of π
(d) y can be a multiple of $\pi/2$

33. If $\cos x + \cos y = a$, $\cos 2x + \cos 2y = b$, $\cos 3x + \cos 3y = c$, then
(a) $\cos^2 x + \cos^2 y = 1 + \frac{b}{2}$
(b) $\cos x \cos y = \frac{a^2}{2} - \left(\frac{b+2}{4}\right)$
(c) $2a^3 + c = 3a(1 + b)$
(d) $a + b + c = 3abc$

34. Consider the system of equations, $x\sin\theta - 2y\cos\theta - az = 0$, $x + 2y + z = 0$, $-x + y + z = 0$, where $\theta \in R$, then
(a) the given system will have infinite solutions for $a = 2$
(b) the number of integer values of a is 3 for the system to have non-trivial solutions.
(c) for $a = 1$ the system will have infinite solutions
(d) for $a = 3$ the system will have unique solution.

35. Which of the following set of values of 'x' satisfies the equation $2^{(2\sin^2 x - 3\sin x + 1)} + 2^{(2 - 2\sin^2 x + 3\sin x)} = 9$
(a) $x = n\pi \pm \frac{\pi}{6}, n \in I$ (b) $x = n\pi \pm \frac{\pi}{3}, n \in I$
(c) $x = n\pi$, $n \in I$ (d) $x = 2n\pi + \frac{\pi}{2}, n \in I$

36. For $0 < \theta < \frac{\pi}{2}$, the solution (s) of
$\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is are
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{12}$ (d) $\frac{5\pi}{12}$

37. If $\tan|x| = |\tan x|$, then x belongs to
(a) $\left(\frac{-3\pi}{2}, -\pi\right]$ (b) $\left[-\pi, \frac{-\pi}{2}\right)$
(c) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{2}, \pi\right]$

SECTION-III

Comprehension Type

Paragraph for Question No. 38 to 40

If $\frac{\cos\theta_1}{\cos\theta_2} + \frac{\sin\theta_1}{\sin\theta_2} = \frac{\cos\theta_0}{\cos\theta_2} + \frac{\sin\theta_0}{\sin\theta_2} = 1$, where θ_1 and θ_0 do not differ by an even multiple of π. Then,

38. $\cos\theta_1 \cos\theta_0$ is equal to
(a) $-\cos^2\theta_2$ (b) $-\cos^4\theta_2$ (c) $-\cos\theta_2$ (d) -1

39. $\sin^2\theta_2$ is equal to
(a) -1 (b) $\sin\theta_1 \sin\theta_0 . \operatorname{cosec}^2\theta_2$
(c) $-\sin\theta_1 \sin\theta_0 \operatorname{cosec}^2\theta_2$ (d) 1

40. $\frac{\cos\theta_1 \cos\theta_0}{\cos^2\theta_2} + \frac{\sin\theta_1 \sin\theta_0}{\sin^2\theta_2}$ is equal to
(a) 1 (b) -1 (c) 0 (d) $-1/2$

Paragraph for Question No. 41 and 42

Minimum and maximum values of $a\cos\theta + b\sin\theta + c$ are $c - \sqrt{a^2 + b^2}$ and $c + \sqrt{a^2 + b^2}$ respectively.

41. If $x^2 + y^2 = x^2y^2$, then range of $\frac{5x + 12y + 7xy}{xy}$ is
(a) [3, 20] (b) [−6, 20] (c) [−6, 3] (d) [7, 20]

42. The range of $3\cos(\theta + 30°) + 3\sin\theta + 2$ is
(a) [−1, 4] (b) [3, 5] (c) [−1, 5] (d) [1,3]

Paragraph for Question No. 43 to 45

In reducing a given equation to the standard forms ($\sin x = \sin\alpha$ etc.), we apply several trigonometric or algebraic transformations. As a result of which the canonical form finally obtained may not be equivalent to the original equation resulting either in loss of solutions or in the appearance of fake solutions.

43. The solution set of the equation $\sqrt{5 - 2\sin x} = 6\sin x - 1$ is
(a) $n\pi + (-1)^n \sin^{-1}\left(-\frac{2}{9}\right)$ or $x = n\pi + (-1)^n \frac{\pi}{6}$
(b) $n\pi + (-1)^n \frac{\pi}{6}$
(c) $n\pi + (-1)^n \sin^{-1}\left(\frac{2}{9}\right)$ (d) null set

44. The equation $2\cot 2x - 3\cot 3x = \tan 2x$ has
(a) two solutions in $(0, \pi/3)$
(b) one solution in $(0, \pi/3)$
(c) no solution in $(-\infty, \infty)$
(d) None of these

45. The solution of the equation $|\cos x| = \cos x - 2\sin x$ is
(a) $n\pi$ or $n\pi + \frac{\pi}{4}$
(b) $n\pi$ if n is odd, $n\pi + \frac{\pi}{4}$ if n is even

(c) $n\pi$ if n is even, $n\pi+\frac{\pi}{4}$ if n is odd
(d) None of these

Paragraph for Question No. 46 to 48

The trigonometric equation is $\sin x + 3\sin 2x + \sin 3x = \cos x + 3\cos 2x + \cos 3x$ when x lies in first four quadrants. It means $x \in [0, 2\pi]$, then

46. How many solutions are there?
(a) 2 (b) 3 (c) 4 (d) 5

47. The difference between greatest and least values of x is
(a) $3\pi/2$ (b) $\pi/2$ (c) π (d) 10π

48. The sum of the values of x is
(a) $\frac{14\pi}{23}$ (b) $\frac{7\pi}{2}$ (c) $\frac{9\pi}{8}$ (d) 6π

Paragraph for Question No. 49 and 50

If $f(x) = g(x)$, then the number of solutions of $f(x) - g(x) = 0$ can be obtained by drawing the graphs of $y = f(x)$ and $y = g(x)$. The number of solutions is equal to the number points, where $y = f(x)$ cuts $y = g(x)$.

49. The number of solutions of $\cos x = 2x^2 + 1$ is
(a) 0 (b) 1 (c) 2 (d) infinite

50. The number of solutions of $\sin x = \frac{|x|}{10}$ is
(a) 4 (b) 6
(c) 8 (d) None of these

SECTION-IV

Matrix-Match Type

51. Let 'P' be an interior point of ΔABC. Match the correct entries for the ratios of the area of ΔPBC to the area of ΔPCA depending on the position of the point 'P' w.r.t ΔABC.

Column-I		Column-II	
(A)	If 'P' is centroid (G)	(p)	$\tan A : \tan B$
(B)	If 'P' is incentre (I)	(q)	$\sin 2A : \sin 2B$
(C)	If 'P' is orthocentre (H)	(r)	$\sin A : \sin B$
(D)	If 'P' is circumcentre (S)	(s)	1 : 1

52. Let

$$f_n(\theta)=\frac{\cos\frac{\theta}{2}+\cos 2\theta+\cos\frac{7\theta}{2}+.....+\cos(3n-2)\frac{\theta}{2}}{\sin\frac{\theta}{2}+\sin 2\theta+\sin\frac{7\theta}{2}+.....+\sin(3n-2)\frac{\theta}{2}}.$$

Then match the entries of Column-I with their corresponding values given in Column-II.

Column-I		Column-II	
(A)	$f_3\left(\frac{3\pi}{16}\right)$	(p)	$2-\sqrt{3}$
(B)	$f_5\left(\frac{\pi}{28}\right)$	(q)	$\sqrt{3+2\sqrt{2}}$
(C)	$f_7\left(\frac{\pi}{60}\right)$	(r)	$\sqrt{2}-1$
		(s)	$\sqrt{7+4\sqrt{3}}$

53. Match the following:

Column-I		Column-II	
(A)	Maximum of $4\sin^2 x+3\cos^2 x+\sin\frac{x}{2}+\cos\frac{x}{2}-\sqrt{2}$ is	(p)	4
(B)	Minimum of $\cos 2\theta+\cos\theta+\frac{9}{8}$ is	(q)	0
(C)	Maximum of $\lvert 3\sin x + 4\cos x - 5\rvert$ is	(r)	10

54. Match the following:

Column-I		Column-II	
(A)	The number of real roots of equation $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ is	(p)	1
(B)	$\sqrt{3}\,\text{cosec}20° - \sec 20° =$	(q)	4
(C)	$4\cos 36° - 4\cos 72° + 4\sin 18° \cos 36° =$	(r)	3

55. Match the following:

	Column-I	Column-II	
(A)	The no. of solutions of the equation $\lvert\cos x\rvert = 2[x]$, (where $[\cdot]$ is g.i.f) is	(p)	8
(B)	The no. of solutions of the equation $2^{\cos x} = \lvert\sin x\rvert$ in $[-2\pi, 2\pi]$ is	(q)	4
(C)	The no. of solutions of the equation $\sin^3 x \cos x + \sin^2 x \cos^2 x + \sin x \cos^3 x = 1$ in $[0, 2\pi]$ are	(r)	0
(D)	The number of possible values of k if fundamental period of $\sin^{-1}(\sin kx)$ is $\pi/2$ is	(s)	2

56. Match the following:

	Column-I	Column-II	
(A)	No. of ordered pairs which satisfy the equation $x^2 + 2x\sin(xy) + 1 = 0$ (where $y \in [0, 2\pi]$) are	(p)	1
(B)	If maximum of $\{5\sin\theta + 3\sin(\theta - \alpha)\}$ is $7, (\theta \in R)$, then no. of solutions of α in $[0, \pi]$ are	(q)	2
(C)	The total number of solutions of $\cos x = \sqrt{1 - \sin 2x}$ in $[0, 2\pi]$ is equal to	(r)	0
(D)	The number of solutions of the equation $\sqrt{\sin x} + 2^{1/4} \cdot \cos x = 0$ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ are	(s)	3

SECTION-V

Integer-Answer Type

57. The value of $\left[(\sqrt{3}+\sqrt{2})\tan\left(7\frac{1}{2}\right)^{\circ}+1\right]^2 =$ ___

58. Suppose $\sin^3 x \sin 3x = \sum_{m=0}^{n} c_m \cos mx$ is an identity in x, where $c_0, c_1, c_2, \ldots, c_n$ are constants and $c_n \neq 0$, then the value of n is

59. If $\tan 20° + \tan 40° + \tan 80° - \tan 60° = \lambda \sin 40°$, then $\lambda =$

60. If $\tan\frac{\pi}{10}$ is a root of polynomial equation $5x^4 - 2kx^2 + 1 = 0$, then the positive integer k must be equal to

61. If in an acute angled triangle ABC, $\tan A$, $\tan B$, $\tan C$, are in H.P., the minimum value of $\cot B$ is $\frac{1}{\sqrt{k}}$, then k is equal to

62. Minimum value of $\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha}$ $\left(\alpha, \beta \neq \frac{k\pi}{2}, k \in Z\right)$ is ___

63. The number of pairs (x, y) of real numbers with $0 < x < \frac{\pi}{2}$ such that

$$\frac{(\sin x)^{2y}}{(\cos x)^{y^2/2}} + \frac{(\cos x)^{2y}}{(\sin x)^{y^2/2}} = \sin 2x \quad \text{is } \_\_\_\_$$

64. If $(0, \pi/n)$ is the first positive interval such that $\tan A < \frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} < \tan C$ for $0 < A < B < C < \frac{\pi}{n}$, then maximum value of n is

65. If $x \in (1, 30)$
$(1 + \tan(x°))(1 + \tan(x + 1)°)(1 + \tan(x + 2)°) \ldots$
$\ldots(1 + \tan(x + 44)°) = 2^{23}$, then value of x is

66. The number of solutions of the equation $(\sqrt{3}\sin x + \cos x)^{\sqrt{(\sqrt{3}\sin 2x + \cos 2x + 2)}} = 4$ in $x \in (-\pi, \pi)$ is

67. The number of solutions of the equation $\sin^{-1}\left(\frac{1+x^2}{2x}\right) = \frac{\pi}{2}\sec(x-1)$ are

68. If m and n are positive integers; $n > m$ then number of solutions of the equation $n\lvert\sin x\rvert = m\lvert\cos x\rvert$ in $[0, 2\pi]$ is

69. Number of solutions of $x \in [0, \pi]$ satisfying the equation $(\log_{\sqrt{3}} \tan x)\left(\sqrt{\log_{\sqrt{3}} 3\sqrt{3} + \log_{\tan x} 3}\right) = -1$ is/are _______

70. The number of solutions of the equation $\sin x + \sin 2x + \sin 3x + \sin 4x = 4$ in the interval $[0, 10\pi]$ is ___

71. The number of solutions of the equation $\sin^4 x - \cos^2 x \sin x + 2\sin^2 x + \sin x = 0$ in $0 \le x \le 3\pi$ is

72. The number of solutions of $2\cos x = |\sin x|$, $0 \le x \le 4\pi$ is

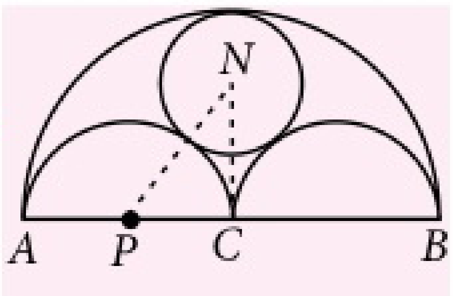
SOLUTIONS

1. (d) : $\because$ A, B, C are in A.P. $\Rightarrow$ $B = \dfrac{\pi}{3}$

$$\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$$

$$= \frac{\sin A}{\sin C}(2\sin C\cos C) + \frac{\sin C}{\sin A}(2\sin A\cos A)$$

$$= 2\sin(A+C) = 2\sin\frac{\pi}{3} = \sqrt{3}$$

2. (c) : 

$PC^2 + NC^2 = PN^2$

$6^2 + (12 - x)^2 = (6 + x)^2 \Rightarrow x = 4$

where x is the radius of required circle.

3. (d) : $s(s-a)(s-b)(s-c) = \Delta^2$,

$r = \dfrac{\Delta}{s} = 1 \Rightarrow s = 6$ units

$\therefore$ $(s - a)(s - b)(s - c) = s$

$s^3 - (a + b + c)s^2 + (ab + bc + ca)s - abc = s$

$\Rightarrow 6^3 - 12(6)^2 + (ab + bc + ca)(6) - 60 = 6$

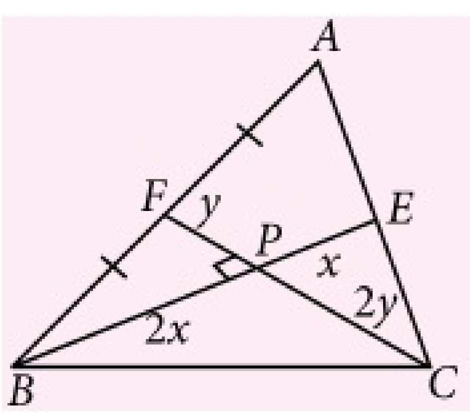
$\Rightarrow ab + bc + ca = 47$ $\therefore$ $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{47}{60} = 0.8$

4. (b) : $x^2 + y^2 = 3 + 2\left(\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{6\pi}{7}\right)$

$$= 3 + 2\left(-\frac{1}{2}\right) = 2$$

5. (a) : $\tan B = \tan\left(\tan^{-1}\left(\dfrac{y}{2x}\right) + \tan^{-1}\left(\dfrac{y}{x}\right)\right)$

$$= \frac{\dfrac{y}{2x} + \dfrac{y}{x}}{1 - \dfrac{y^2}{2x^2}} = \frac{3xy}{2x^2 - y^2}$$



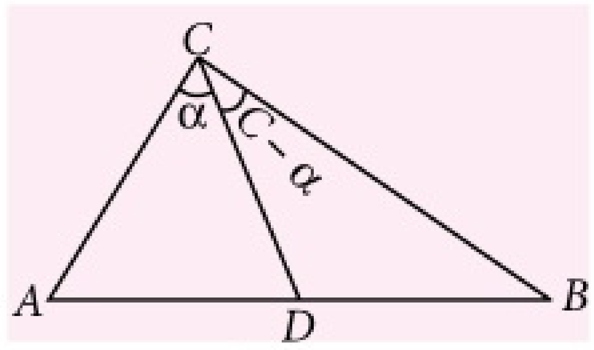
$\tan\angle PBF = \dfrac{y}{2x}$

$\tan\angle PBC = \dfrac{y}{x}$

$\Rightarrow \cot B = \dfrac{2x^2 - y^2}{3xy}$, $\cot C = \dfrac{2y^2 - x^2}{3xy}$

$\Rightarrow \cot B + \cot C = \dfrac{x^2 + y^2}{3xy} \ge \dfrac{2}{3}$

6. (b) : Let $\angle ACD = \alpha$ $\Rightarrow$ $\angle DCB = C - \alpha$



Applying the sine rule in ΔACD and in ΔDCB respectively, we get

$$\frac{AD}{\sin\alpha} = \frac{CD}{\sin A} \text{ and } \frac{BD}{\sin(C-\alpha)} = \frac{CD}{\sin B}$$

$$\Rightarrow \frac{AD \cdot BD}{\sin\alpha \cdot \sin(C-\alpha)} = \frac{CD^2}{\sin A \cdot \sin B}$$

$$\Rightarrow \frac{1}{2}[\cos(2\alpha - C) - \cos C] = \sin A \sin B$$

$$\Rightarrow \frac{1}{2}\left[\cos(2\alpha - C) - 1 + 2\sin^2\frac{C}{2}\right] = \sin A \sin B$$

$$\Rightarrow \sin^2\frac{C}{2} - \frac{1}{2}(1 - \cos(2\alpha - C)) = \sin A \sin B$$

Since, $1 - \cos(2\alpha - C) \ge 0$

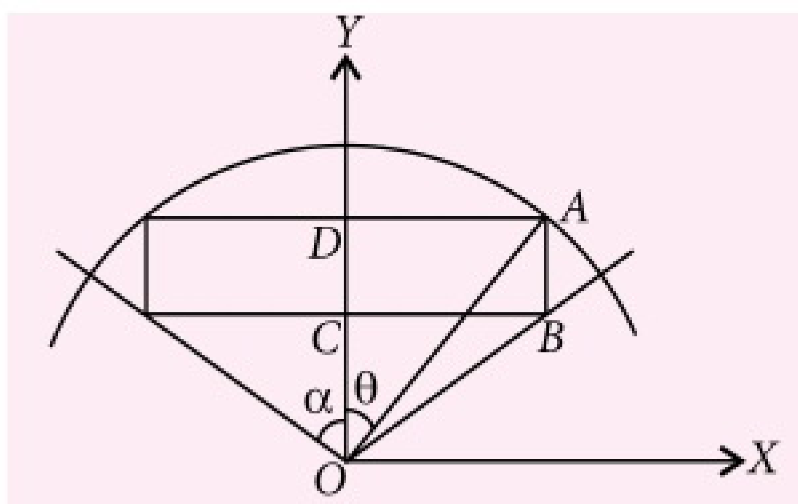
$$\Rightarrow \sin A \cdot \sin B \le \sin^2\frac{C}{2}$$

and equality sign holds, if $1 - \cos(2\alpha - C) = 0$

$$\Rightarrow \alpha = \frac{C}{2}$$

That means equality sign holds, if CD is the internal bisector of $\angle C$.

7. (b) : Let A be any point on the arc such that $\angle YOA = \theta$, where $0 \le \theta \le \alpha$.



$DA = CB = R\sin\theta$, $OD = R\cos\theta$

$\Rightarrow CO = CB\cot\alpha = R\sin\theta\cot\alpha$

Now, $CD = OD - OC = R\cos\theta - R\sin\theta\cot\alpha$

$= R(\cos\theta - \sin\theta\cot\alpha)$

Area of rectangle $ABCD$, $S = CD\cdot CB$

$= R(\cos\theta - \sin\theta\cot\alpha)R\sin\theta$

$= R^2(\sin\theta\cos\theta - \sin^2\theta\cot\alpha)$

$$= \frac{R^2}{2}(\sin 2\theta - (1-\cos 2\theta)\cot\alpha)$$

$$= \frac{R^2}{2\sin\alpha}[\cos(2\theta-\alpha)-\cos\alpha]$$

$$\Rightarrow S_{max} = \frac{R^2}{2\sin\alpha}(1-\cos\alpha)\left(\text{for } \theta = \frac{\alpha}{2}\right)$$

$$= \frac{R^2}{2}\left(\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) = \frac{R^2}{2}\tan\frac{\alpha}{2}$$

Hence, greatest area of the rectangle $= \dfrac{R^2}{2}\tan\dfrac{\alpha}{2}$

8. (c) : Let $f(x) = \sin(\cos x) + \cos(\sin x)$

$\Rightarrow f'(x) = \cos(\cos x)(-\sin x) - \sin(\sin x)\cos x$

$\Rightarrow f'(x) \le 0$ for $x \in \left[0, \frac{\pi}{2}\right]$

$\Rightarrow$ 'f' is decreasing function

$\therefore$ maximum value $= f(0) = \sin 1 + 1$

Since 'f' is an even function its enough to check on $[0, \pi/2]$

$\therefore$ Minimum value $= f(\pi/2) = 0 + \cos 1 = \cos 1$

9. (c) : $g(x) = \frac{1}{2}\tan^{-1}|x| + 1 \Rightarrow \text{sgn}(g(x)) = 1$

$\Rightarrow \sin^{23}x - \cos^{22}x = 1$

$\Rightarrow \sin^{23}x = 1 + \cos^{22}x$ which is possible only if $\sin x = 1$ and $\cos x = 0$

$$\Rightarrow x = 2n\pi + \frac{\pi}{2}$$

hence $-10\pi \le 2n\pi + \frac{\pi}{2} \le 20\pi$

$$\Rightarrow \frac{-21\pi}{2} \le 2n\pi \le \frac{39\pi}{2} \Rightarrow \frac{-21}{4} \le n \le \frac{39}{4} \Rightarrow -5 \le n \le 9$$

Hence no. of values of $x = 15$.

10. (d) : (a) $\cos(\tan^{-1}(\tan(4-\pi))) = \cos(4-\pi)$
$= -\cos 4 > 0$

(b) $\sin(\cot^{-1}(\cot(4-\pi))) = \sin(4-\pi) = -\sin 4 > 0$

(c) $\tan(\cos^{-1}(\cos(2\pi - 5))) = \tan(2\pi - 5)$
$= -\tan 5 > 0$

(d) $\cot(\sin^{-1}(\sin(\pi - 4))) = \cot(\pi - 4) = -\cot 4 < 0$

11. (b) : $2P_6 - 3P_4$

$= 2(\cos^6\theta + \sin^6\theta) - 3(\cos^4\theta + \sin^4\theta)$

$= 2(1 - 3\sin^2\theta\cos^2\theta) - 3(1 - 2\sin^2\theta\cos^2\theta) = -1$

12. (a) : $\cos A + \cos B + \cos C = 1 + \frac{r}{R} = \frac{9}{8}$

$$\Rightarrow \left(4\sin\frac{C}{2} - 1\right)^2 = 0 \Rightarrow \sin\frac{C}{2} = \frac{1}{4}$$

13. (c) : $2|\sin 2\alpha| \le 2$ and

$|\tan\beta + \cot\beta| = |\tan\beta| + |\cot\beta| \ge 2$

$$\Rightarrow \alpha = \beta = \frac{3\pi}{4}$$

14. (b) : Applying $C_1 \to C_1 + C_2 + C_3$ gives

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ gives

$$(\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & (\sin x - \cos x) & 0 \\ 0 & 0 & (\sin x - \cos x) \end{vmatrix} = 0$$

$\Rightarrow (\sin x + 2\cos x)(\sin x - \cos x)^2 = 0$

$$\Rightarrow \tan x = -2,\ \tan x = 1 \Rightarrow x = \frac{\pi}{4}$$

15. (a) : For non-trivial solution, we have

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0 \Rightarrow \sin 3\theta + 2\cos\theta - 2 = 0$$

$\Rightarrow (3\sin\theta - 4\sin^3\theta) + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow 4\sin^3\theta + 4\sin^2\theta - 3\sin\theta = 0$

$\Rightarrow \sin\theta(2\sin\theta - 1)(2\sin\theta + 3) = 0$

$$\Rightarrow \text{either } \sin\theta = 0 \text{ or } \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = n\pi + (-1)^n\frac{\pi}{6}, \forall n \in Z$$

16. (b) : $3^{\sin 2x + 2\cos^2 x} + 3^{1+2(1-\cos^2 x) - \sin 2x} = 28$

$$\Rightarrow 3^{\sin 2x + 2\cos^2 x} + \frac{27}{3^{\sin 2x + 2\cos^2 x}} = 28$$

Put $\sin^2 x + 2\cos^2 x = t$, we get

$(3^t)^2 - 28(3^t) + 27 = 0 \Rightarrow 3^t = 27$ or 1

$\Rightarrow \sin 2x + 2\cos^2 x = 0$ or 3

$\Rightarrow$ sin2x + 2cos^2x = 0 $\Rightarrow$ sin2x + cos2x = –1
$\Rightarrow$ either sin2x = –1 or cos2x = –1

either $2x = \frac{3\pi}{2}, \frac{7\pi}{2}$, or $2x = \pi$ or 3π

$$\Rightarrow \quad x = \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{2}$$

17. (b) : $k = \frac{2\tan x}{1+\tan^2 x} = \sin 2x \;\; (\because 0 < k < 1)$

$$\Rightarrow \quad 2A = \pi - 2B \Rightarrow C = \frac{\pi}{2} \text{ and } \sin C = 1$$

18. (b) : a, b, c are roots of equation
$x\sin\theta + y\sin2\theta + z\sin3\theta = \sin4\theta$
$\Rightarrow$ $x\sin\theta + y(2\sin\theta\cos\theta) + z(3\sin\theta - 4\sin^3\theta)$
$= 4\sin\theta\cos\theta\cos2\theta$

$$\Rightarrow \cos^3\theta - \frac{z}{2}\cos^2\theta - \frac{y+2}{4}\cos\theta + \frac{z-x}{8} = 0$$

19. (c) : $f(\theta) = \sin\theta(\sin\theta + \sin3\theta)$
$= (\sin\theta + 3\sin\theta - 4\sin^3\theta)\sin\theta$
$= (4\sin\theta - 4\sin^3\theta)\sin\theta = \sin^2\theta(4 - 4\sin^2\theta)$
$= 4\sin^2\theta(1 - \sin^2\theta) = 4\sin^2\theta\cos^2\theta$
$= (2\sin\theta\cos\theta)^2 = (\sin2\theta)^2 \geq 0$
which is true for all θ.

20. (b) : $\frac{\sin^2 2x + 4\sin^4 x - 4\sin^2 x\cos^2 x}{4 - 4\sin^2 x - \sin^2 2x} = \frac{1}{9}$

$$\Rightarrow \quad \frac{4\sin^4 x}{4\cos^2 x - 4\sin^2 x\cos^2 x} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{\sin^4 x}{\cos^4 x} = \frac{1}{9} \Rightarrow \tan x = \pm\frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6}$$

21. (d) : $|x| + |y| = 2 \Rightarrow |x|, |y| \in [0, 2]$

Also $\sin\left(\frac{\pi x^2}{3}\right) = 1 \Rightarrow \frac{\pi x^2}{3} = (4n+1)\frac{\pi}{2}$

$$\Rightarrow \quad x^2 = (4n+1)\frac{3}{2}$$

$\because$ $|x| \in [0, 2]$, then only possible value of $x^2 = 3/2$

$\therefore |x| = \sqrt{\frac{3}{2}}, \; |y| = 2 - \sqrt{\frac{3}{2}}$

Hence, total number of ordered pairs is 4.

22. (b) : The solution is only possible, when

$\cos^2\left(\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right) = 1$ .... (1)

and $\tan^2\left(x + \frac{\pi}{4}\tan^2 x\right) = 0$ ..... (2)

Let's solve (1) first

$$\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x) = k\pi$$

$\Rightarrow$ $\sin x + \sqrt{2}\cos^2 x = 4k$ [only possible for $k = 0$]

$$\Rightarrow \quad \sin x + \sqrt{2}\cos^2 x = 0$$

$$\Rightarrow \sqrt{2}\sin^2 x - \sin x - \sqrt{2} = 0 \Rightarrow \sin x = \frac{-1}{\sqrt{2}}, \sqrt{2}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{4} \text{ or } 2n\pi + \frac{5\pi}{4}$$

From equation (2), we can say that the only solution possible is $x = 2n\pi - \frac{\pi}{4}$

Hence for $x \in [-2\pi, 2\pi]$, we have 2 solutions.

23. (c) : $x^2 + 5 + 4\cos(\alpha x + \beta) = 2x$
$\Rightarrow$ $(x - 1)^2 + 4 = -4\cos(\alpha x + \beta)$
Here, $(x - 1)^2 + 4 \geq 4$, $-4\cos(\alpha x + \beta) \leq 4$
But, the given equation has atleast one solution, then
$(x - 1)^2 + 4 = 4 = -4\cos(\alpha x + \beta)$
$\therefore$ $x = 1 \Rightarrow \cos(\alpha + \beta) = -1$
$\Rightarrow$ $\alpha + \beta = 3\pi$ $(\because \alpha + \beta \in [4, 10])$

24. (d) : $\sin8\theta + \sin2\theta = \sin16\theta + \sin2\theta$
or $\sin16\theta = \sin8\theta$

$$\therefore \quad \theta = \frac{m\pi}{4}, \frac{(2m+1)\pi}{24}, \text{ when } m \in Z$$

$$= 0, \frac{\pi}{4}, \frac{\pi}{2} \text{ and } \frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{7\pi}{24}, \frac{3\pi}{8}, \frac{11\pi}{24}$$

25. (c) : $|\sqrt{m} - \sqrt{n}| \leq \sqrt{|m-n|}$

$$\Rightarrow \left|\sqrt{\sin^2 x + 2a^2} - \sqrt{2a^2 - 1 - \cos^2 x}\right|$$

$$\leq \sqrt{\sin^2 x + 2a^2 - 2a^2 + 1 + \cos^2 x} \leq \sqrt{2}$$

26. (b, d) : $2p = \cos(A - B - C + D) - \cos(A - B + C - D)$
$\Rightarrow$ $2q = \cos(B - C - A + D) - \cos(B - C + A - D)$
$\Rightarrow$ $2r = \cos(C - A - B + D) - \cos(C - A + B - D)$
$\Rightarrow$ $2p + 2q + 2r = 0$
$\Rightarrow$ $p + q + r = 0$ and $p^3 + q^3 + r^3 = 3pqr$

27. (b, d) : $(a + 2)\tan A + (2a - 1) = (2a + 1)\sec A$
Squaring on both sides and simplifying,

28. (a, b, c) : $\sin x + \sin y = \frac{1}{4}$ ... (1)

$\cos x + \cos y = \frac{1}{3}$ ... (2)

$$\frac{(1)}{(2)} \Rightarrow \frac{2\sin\left(\frac{x+y}{2}\right).\cos\left(\frac{x-y}{2}\right)}{2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)}=\frac{3}{4}$$

$$\Rightarrow \tan\left(\frac{x+y}{2}\right)=\frac{3}{4}$$

Now,

$$\tan(x+y)=\frac{2\tan\left(\frac{x+y}{2}\right)}{1-\tan^2\left(\frac{x+y}{2}\right)}=\frac{2\times\frac{3}{4}}{1-\frac{9}{16}}=\frac{6}{4}\times\frac{16}{7}=\frac{24}{7}$$

$$\Rightarrow \cos(x+y)=\frac{7}{25}$$

29. (a, c) : $\frac{a}{\sin A}=\frac{b}{\sin B}$ or $a\sin B=b\sin A$

(a) $b\sin A = a \Rightarrow \sin B = 1 \Rightarrow \angle B=\frac{\pi}{2}$

$\Rightarrow \angle A<\frac{\pi}{2}$, then ΔABC is possible

(b) $b\sin A > a \Rightarrow \sin B > 1$ (impossible)

(c) $b\sin A < a \Rightarrow a\sin B < a \Rightarrow \sin B < 1 \Rightarrow \angle B$ exists

Since $b > a \Rightarrow \angle B > \angle A$

Since $\angle A<\frac{\pi}{2} \Rightarrow \angle B<\frac{\pi}{2}$ or $>\frac{\pi}{2}$

$\Rightarrow \Delta ABC$ is possible.

(d) $b\sin A > a \Rightarrow a\sin B > a \Rightarrow \sin B > 1$ (impossible)

30. (a, c) : $a^2 = b^2 + c^2 - 2bc\cos A$

$\Rightarrow c^2 - (2b\cos A)c + b^2 - a^2 = 0$

$\Rightarrow c_1 + c_2 = 2b\cos A,\ c_1c_2 = b^2 - a^2$

31. (b, d) : $\angle B = 60° \Rightarrow \cos B = 1/2$

$\Rightarrow a^2 + c^2 - b^2 = ac$

32. (a, d) : $\left(\cos^2 x+\frac{1}{\cos^2 x}\right)(1+\tan^2 2y)(3+\sin 3z)=4$

Since, $\cos^2 x+\frac{1}{\cos^2 x}\geq 2,\ 1+\tan^2 2y\geq 1,$

$2 \leq 3 + \sin 3z \leq 4$

So, the only possibility is

$\cos^2 x+\frac{1}{\cos^2 x}=2,\ 1 + \tan^2 2y = 1,\ 3 + \sin 3z = 2$

$\Rightarrow \cos x = \pm 1 \Rightarrow x = n\pi,$

$\tan 2y=0\Rightarrow y=\frac{m\pi}{2}$ and

$\sin 3z=-1 \Rightarrow z=(4k-1)\frac{\pi}{6};\ m,n,k\in I$

33. (a, b, c) : $(\cos x + \cos y)^2 = a^2$

$\Rightarrow \cos^2 x + \cos^2 y + 2\cos x\cos y = a^2$ ... (1)

$\cos 2x + \cos 2y = b$

$\Rightarrow 2\cos^2 x - 1 + 2\cos^2 y - 1 = b$

$\Rightarrow 2[\cos^2 x + \cos^2 y] = b + 2$ ... (2)

$\Rightarrow \cos^2 x + \cos^2 y = \frac{b}{2}+1$

From (1) and (2),

$$2\cos x\cos y=a^2-\left(\frac{b+2}{2}\right)$$

$$\Rightarrow \cos x\cos y=\frac{a^2}{2}-\left(\frac{b+2}{4}\right)$$

$\cos 3x + \cos 3y = c$

$\Rightarrow 4\cos^3 x - 3\cos x + 4\cos^3 y - 3\cos y = c$

$\Rightarrow 4[\cos^3 x + \cos^3 y] - 3[\cos x + \cos y] = c$

$\Rightarrow 4[(\cos x + \cos y)(\cos^2 x + \cos^2 y - \cos x\cos y)]$
$- 3(\cos x + \cos y) = c$

$$\Rightarrow 4\left[a\left(\frac{b+2}{2}-\frac{1}{2}\left(a^2-\frac{b+2}{2}\right)\right)\right]-3a=c$$

$\Rightarrow 2ab + 4a - 2a^3 + ab + 2a = 3a + c$

$\Rightarrow 2a^3 + c = 3a(1 + b)$

34. (b, c, d) : For the system to have non-trivial solution, we have $\begin{vmatrix}\sin\theta & -2\cos\theta & -a\\ -1 & 1 & 1\\ 1 & 2 & 1\end{vmatrix}=0$

i.e. $\sin\theta + 4\cos\theta = 3a \Rightarrow -\frac{\sqrt{17}}{3}\leq a\leq\frac{\sqrt{17}}{3}$

$\Rightarrow a$ has three integer values.

35. (a, d) : $2^{(2\sin^2 x-3\sin x+1)}+2^{3-(2\sin^2 x-3\sin x+1)}=9$

Let $2^{(2\sin^2 x - 3\sin x + 1)} = t$

$\Rightarrow t+\frac{8}{t}=9\Rightarrow t^2-9t+8=0\Rightarrow t = 1, 8$

$\Rightarrow 2\sin^2 x - 3\sin x + 1 = 3$ or $2\sin^2 x - 3\sin x + 1 = 0$

$\Rightarrow \sin x=-\frac{1}{2}, \sin x=\frac{1}{2}, \sin x=1$

36. (c, d) : We have,

$$\sum_{m=1}^{6}\text{cosec}\left[\theta+\frac{(m-1)\pi}{4}\right]\text{cosec}\left[\theta+\frac{m\pi}{4}\right]=4\sqrt{2}$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\sin\frac{\pi}{4}}{\sin\left[\theta+\frac{(m-1)\pi}{4}\right]\sin\left[\theta+\frac{m\pi}{4}\right]}=4$$

$$\Rightarrow \sum_{m=1}^{6} \frac{\sin\left[\left(\theta+\frac{m\pi}{4}\right)-\left(\theta+\frac{(m-1)\pi}{4}\right)\right]}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)}=4$$

$$\Rightarrow \sum_{m=1}^{6}\left[\frac{\sin\left(\theta+\frac{m\pi}{4}\right)\cos\left(\theta+\frac{(m-1)\pi}{4}\right)-\cos\left(\theta+\frac{m\pi}{4}\right)\sin\left(\theta+\frac{(m-1)\pi}{4}\right)}{\sin\left(\theta+\frac{(m-1)\pi}{4}\right)\sin\left(\theta+\frac{m\pi}{4}\right)}\right]=4$$

$$\Rightarrow \sum_{m=1}^{6}\left[\cot\left(\theta+\frac{(m-1)\pi}{4}\right)-\cot\left(\theta+\frac{m\pi}{4}\right)\right]=4$$

$$\Rightarrow\left[\cot\theta-\cot\left(\theta+\frac{\pi}{4}\right)\right]+\left[\cot\left(\theta+\frac{\pi}{4}\right)-\cot\left(\theta+\frac{2\pi}{4}\right)\right]+\dots$$

$$\dots+\left[\cot\left(\theta+\frac{5\pi}{4}\right)-\cot\left(\theta+\frac{6\pi}{4}\right)\right]=4$$

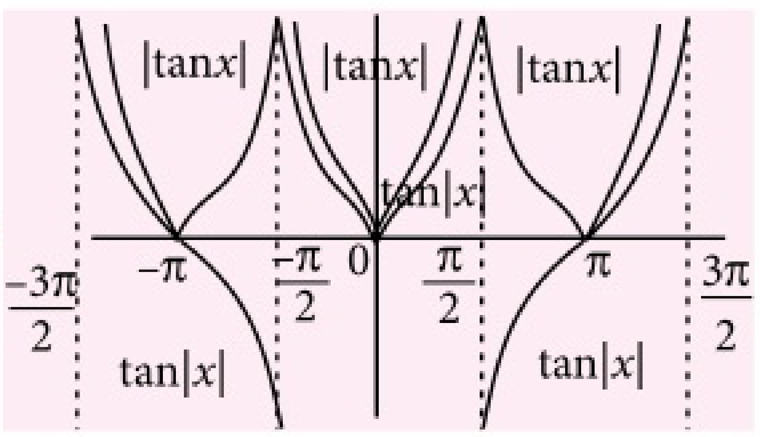
$$\Rightarrow \cot\theta-\cot\left(\theta+\frac{3\pi}{2}\right)=4\Rightarrow \cot\theta+\tan\theta=4$$

$\Rightarrow\ \cos^2\theta + \sin^2\theta = 4\sin\theta\cos\theta$

$$\Rightarrow \sin 2\theta=\frac{1}{2}\ \Rightarrow\ 2\theta=\frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\Rightarrow \theta=\frac{\pi}{12} \text{ or } \frac{5\pi}{12}$$

37. (a, c) :



(38 - 40) :

38. (b) **39. (c)** **40. (b)**

Clearly θ_1 and θ_0 lie on $\dfrac{\cos\theta}{\cos\theta_2}+\dfrac{\sin\theta}{\sin\theta_2}=1$

$$\Rightarrow \left(\frac{\sin\theta}{\sin\theta_2}\right)^2=\left(1-\frac{\cos\theta}{\cos\theta_2}\right)^2$$

$$\Rightarrow \frac{\sin^2\theta}{\sin^2\theta_2}=1+\frac{\cos^2\theta}{\cos^2\theta_2}-\frac{2\cos\theta}{\cos\theta_2}$$

$$\Rightarrow \frac{1-\cos^2\theta}{\sin^2\theta_2}=1+\frac{\cos^2\theta}{\cos^2\theta_2}-\frac{2\cos\theta}{\cos\theta_2}$$

$$\Rightarrow \left(\frac{1}{\cos^2\theta_2}+\frac{1}{\sin^2\theta_2}\right)\cos^2\theta-\frac{2}{\cos\theta_2}\cos\theta+1-\frac{1}{\sin^2\theta_2}=0$$

which is a quadratic equation in $\cos\theta$.
It has two roots $\cos\theta_0$, $\cos\theta_1$

$$\Rightarrow \cos\theta_1.\cos\theta_0=\frac{1-\frac{1}{\sin^2\theta_2}}{\frac{1}{\cos^2\theta_2}+\frac{1}{\sin^2\theta_2}}$$

$$=\frac{-\cos^2\theta_2}{\frac{1}{\cos^2\theta_2}}=-\cos^4\theta_2 \quad \dots(1)$$

Similarly, $$\left(\frac{1}{\cos^2\theta_2}+\frac{1}{\sin^2\theta_2}\right)\sin^2\theta-\frac{2}{\sin\theta_2}\sin\theta+1-\frac{1}{\cos^2\theta_2}=0$$

$\Rightarrow\ \sin\theta_1\sin\theta_0 = -\sin^4\theta_2$ ...(2)

$\therefore$ From (1) and (2)

$$\frac{\cos\theta_1\cos\theta_0}{\cos^2\theta_2}+\frac{\sin\theta_1.\sin\theta_0}{\sin^2\theta_2}=-\cos^2\theta_2-\sin^2\theta_2=-1$$

41. (b) : Let $x = \sec\theta$, $y = \text{cosec}\theta$

$\Rightarrow\ x^2 + y^2 = x^2y^2$

$$\therefore\ \frac{5x+12y+7xy}{xy}=5\sin\theta+12\cos\theta+7$$

42. (c) : $3\cos(\theta + 30°) + 3\sin\theta + 2$

$$=\frac{3\sqrt{3}}{2}\cos\theta+\frac{3}{2}\sin\theta+2$$

$$\text{Range}=\left[2-\sqrt{\frac{27+9}{4}},2+\sqrt{\frac{27+9}{4}}\right]$$

$= [2 - 3, 2 + 3] = [-1, 5]$

43. (b) : The equation has a meaning if $\sin x \le \frac{5}{2}$, which is always true. Any x for which $\sin x < \frac{1}{6}$ cannot be solution, since $\sqrt{5-2\sin x} > 0$ for all x. On squaring the equation and solving, we get $\sin x = -\frac{2}{9}$ or $\frac{1}{2}$. But $\sin x = -\frac{2}{9}$ is not possible

$$\left(\because \sin x < \frac{1}{6} \text{ is not possible}\right)$$

$\Rightarrow \sin x = \frac{1}{2}$

44. (c) : The equation has a meaning if

$$2x \ne n\pi, 3x \ne n\pi, 2x \ne (2n+1)\frac{\pi}{2}$$

For these values, the equation can be written as,

$$3\left(\frac{\cos 2x}{\sin 2x} - \frac{\cos 3x}{\sin 3x}\right) = \frac{\sin 2x}{\cos 2x} + \frac{\cos 2x}{\sin 2x}$$

$$\Rightarrow \frac{3\sin x}{\sin 2x \sin 3x} = \frac{1}{\sin 2x \cos 2x}$$

$\Rightarrow$ $3\sin x\cos 2x = \sin 3x$

$\Rightarrow$ $\sin x(3 - 4\sin^2 x - 3\cos 2x) = 0$

$\Rightarrow$ $\sin x \sin^2 x = 0$

$\Rightarrow$ $\sin x = 0 \Rightarrow x = n\pi$

But for $x = n\pi$, the equation has no meaning.

$\Rightarrow$ The given equation has no solution.

45. (c): If $\cos x \ge 0$, then the equation is equivalent to $\cos x = \cos x - 2\sin x$

$\Rightarrow$ $\sin x = 0 \Rightarrow x = n\pi$

But in order to have $\cos x > 0$, n should be even.

Again if $\cos x < 0$ then the equation is equivalent to $\tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}$

But in order to have $\cos x < 0$, we must choose n odd.

Thus the correct solution set is

$n\pi$ (n even) or $n\pi + \frac{\pi}{4}$ (n odd).

46. (c) : Apply $\sin C + \sin D$ and $\cos C + \cos D$, then take common and solve, we get

$(\sin 2x - \cos 2x)(2\cos x + 3) = 0$

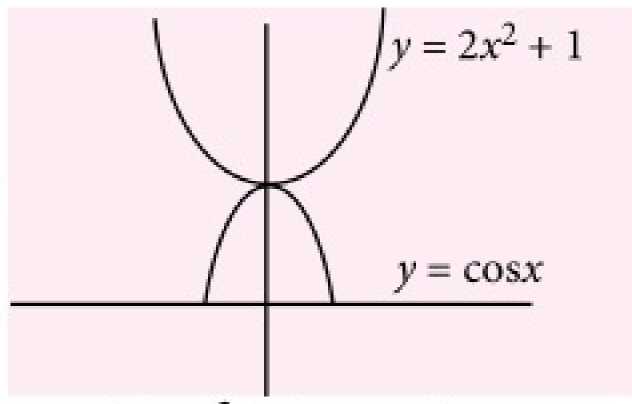
$\Rightarrow \tan 2x = 1, \cos x = \frac{-3}{2}$ (Rejected)

$\Rightarrow x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

47. (a) : $\frac{13\pi}{8} - \frac{\pi}{8} = \frac{12\pi}{8} = \frac{3\pi}{2}$

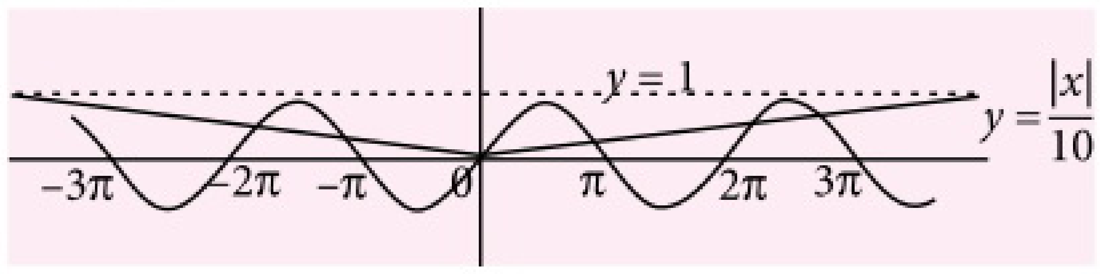
48. (b) : $\frac{28\pi}{8} = \frac{7\pi}{2}$

49. (b) :



Only one point of intersection

50. (b) :

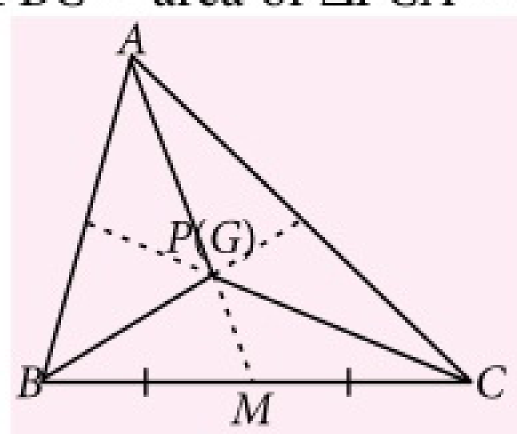


$\therefore$ Number of solutions = 6

51. (A – s; B – r; C – p; D – q)

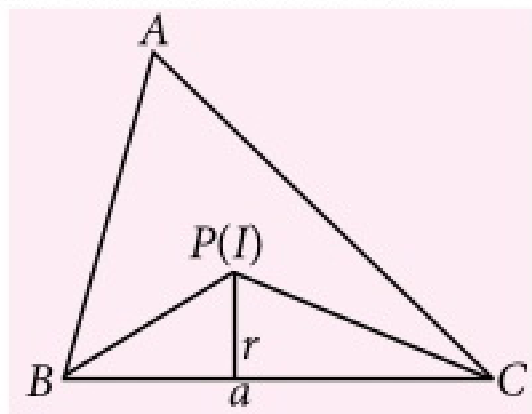
(A) Using properties of median

$\therefore$ area of ΔPBC = area of ΔPCA = area of ΔPAB



$\therefore$ Required ratio = 1 : 1

(B) area of ΔPBC : area of ΔPCA

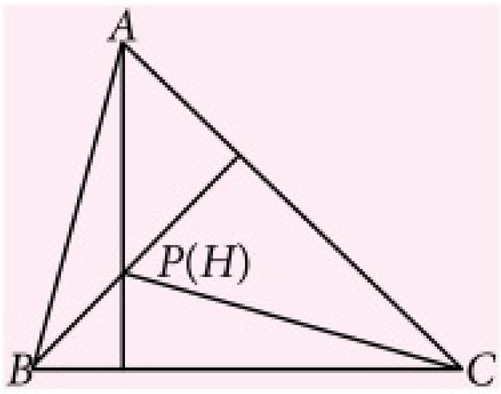


$= \frac{1}{2}ar : \frac{1}{2}br = a : b = \sin A : \sin B$

(C) area of ΔPBC : area of ΔPCA

$= \frac{1}{2}a(2R\cos B\cos C) : \frac{1}{2}b(2R\cos C\cos A)$

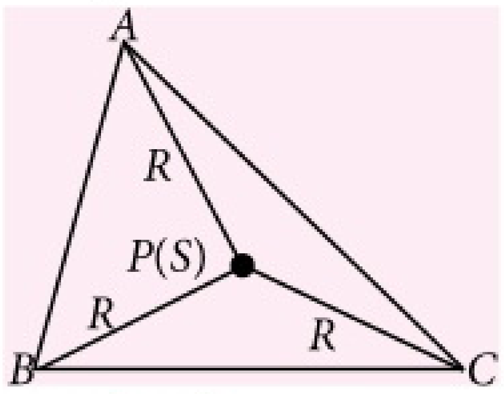
$= \sin A\cos B\cos C : \sin B\cos C\cos A$



On dividing by $\prod \cos A$, we get $\tan A : \tan B$

(D) area of ΔPBC : area of ΔPCA

$$= \frac{1}{2}R^2 \sin 2A : \frac{1}{2}R^2 \sin 2B = \sin 2A : \sin 2B$$



52. (A – r; B – q; C – s)

We have, $f_n(\theta) = \cot\left(\left(3n-1\right)\frac{\theta}{4}\right)$

(A) $f_3\left(\frac{3\pi}{16}\right) = \cot\left(8 \times \frac{1}{4} \times \frac{3\pi}{16}\right) = \cot\frac{3\pi}{8} = \sqrt{2}-1$

(B) $f_5\left(\frac{\pi}{28}\right) = \cot\left(14 \times \frac{1}{4} \times \frac{\pi}{28}\right) = \cot\frac{\pi}{8}$

$$= \sqrt{2}+1 = \sqrt{3+2\sqrt{2}}$$

(C) $f_7\left(\frac{\pi}{60}\right) = \cot\left(20 \times \frac{1}{4} \times \frac{\pi}{60}\right) = \cot\frac{\pi}{12}$

$$= (2+\sqrt{3}) = \sqrt{7+4\sqrt{3}}$$

53. (A - p; B - q; C - r)

(A) $\frac{4(1-\cos 2x)}{2} + \frac{3(1+\cos 2x)}{2} + \sqrt{1+\sin x}$

$= \frac{7-\cos 2x}{2} + \sqrt{1+\sin x}$

$= 3 + \sin^2 x + \sqrt{1+\sin x} = 4 + \sqrt{2}$ (Maximum)

(B) $\cos 2\theta + \cos\theta = \frac{-9}{8} + 2\left(\cos\theta + \frac{1}{4}\right)^2$

$\Rightarrow$ Minimum $= \frac{-9}{8}$

(C) $-10 \le 3\sin x + 4\cos x - 5 \le 0$

$\Rightarrow \max|3\sin x + 4\cos x - 5| = 10$

54. (A – r; B – q; C – r)

(A) $\cos^7 x + \sin^4 x = 1$

$\Rightarrow \cos^7 x = 1, \sin^4 x = 0$ or $\cos^7 x = 0, \sin^4 x = 1$

$\therefore\ x = 0$ or $\pm\frac{\pi}{2}$

(B) $\frac{\sqrt{3}\cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$

$= \frac{4(\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ)}{2\sin 20^\circ \cos 20^\circ}$

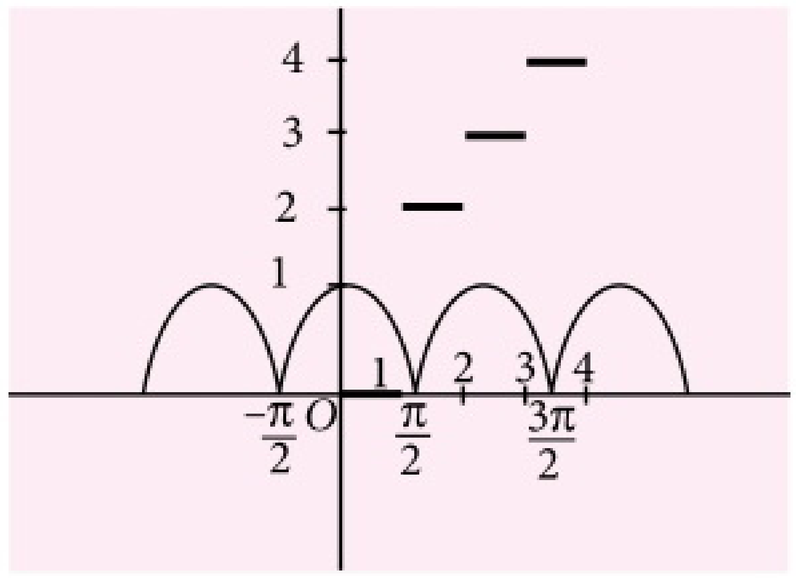
$= \frac{4\sin 40^\circ}{\sin 40^\circ} = 4$

(C) $4\left(\frac{\sqrt{5}+1}{4}\right) - 4\left(\frac{\sqrt{5}-1}{4}\right) + 4\left(\frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4}\right)$

$= 2 + 1 = 3$

55. (A - r; B - p; C - r; D - s)

(A) $y = |\cos x|$, $y = 2[x]$



(B) The number of solutions is 8

(C) $\sin^3 x\cos x + \sin^2 x\cos^2 x + \sin x\cos^3 x = 1$

$\Rightarrow \sin x\cos x(\sin^2 x + \sin x\cos x + \cos^2 x) = 1$

$\Rightarrow \frac{\sin 2x}{2}\left(1 + \frac{\sin 2x}{2}\right) = 1$

$\Rightarrow \sin 2x(2 + \sin 2x) = 4 \Rightarrow \sin^2 2x + 2\sin 2x - 4 = 0$

$\Rightarrow \sin 2x = \frac{-2 \pm \sqrt{4+16}}{2} = -1 \pm \sqrt{5}$ (Impossible)

(D) $\frac{2\pi}{|k|} = \frac{\pi}{2} \Rightarrow |k| = 4$

56. (A - q; B - p; C - q; D - r)

(A) $x^2 + 2x\sin(xy) + 1 = 0$

$\Rightarrow [x + \sin(xy)]^2 + 1 - \sin^2(xy) = 0$

$\Rightarrow x + \sin(xy) = 0$ and $\sin^2(xy) = 1$

If $\sin^2(xy) = 1 \Rightarrow \sin(xy) = 1$ or -1

If $\sin(xy) = 1$. Let $x = -1 \Rightarrow \sin(-y) = 1$

$\Rightarrow y = 3\pi/2 \ \therefore (x, y) = (-1, 3\pi/2)$

If $\sin(xy) = -1$. Let $x = 1 \Rightarrow \sin y = -1$

$\therefore (x, y) = (1, 3\pi/2)$

(B) $5\sin\theta + 3(\sin\theta\cos\alpha - \cos\theta\sin\alpha)$

$= (5 + 3\cos\alpha)\sin\theta - 3\sin\alpha\cos\theta$

Max. value $= \sqrt{(5+3\cos\alpha)^2 + 9\sin^2\alpha}$

$= \sqrt{34+30\cos\alpha}$

$\therefore$ 34 + 30cosα = 49

$\Rightarrow$ cosα = 1/2

(C) $\cos x = |\sin x - \cos x|$

i) If $\sin x < \cos x \Rightarrow x \in \left[0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right]$

$\Rightarrow \sin x = 0 \quad \Rightarrow \quad x = 2\pi$

ii) If $\sin x \geq \cos x \Rightarrow x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\therefore \tan x = 2$ or $x = \tan^{-1}2$

(D) $\sqrt{\sin x} + 2^{1/4} \cdot \cos x = 0$

Clearly $\cos x < 0$ $\therefore$ x lies in 2nd or 3rd Quadrant

$2^{1/4}\cos x = -\sqrt{\sin x}; \ \sqrt{2}\cos^2 x = \sin x$

$\Rightarrow \sin x = \frac{1}{\sqrt{2}}, \quad \because x \in Q_2 \ \therefore x = \frac{3\pi}{4}$

57. (2) : $[(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)+1]^2$

$= [\sqrt{2}]^2 = 2$

58. (6) : $\sin^3 x \cdot \sin 3x = \sin^2 x \cdot \sin x \cdot \sin 3x$

$= \frac{(1-\cos 2x)}{2} \cdot \frac{(\cos 2x - \cos 4x)}{2}$

$= \frac{-1}{8} + \frac{3}{8}\cos 2x - \frac{3}{8}\cos 4x + \frac{1}{8}\cos 6x$

59. (8) : (tan20° + tan80°) – (tan60° – tan40°)

$= \frac{\sin 100°}{\cos 20° \cos 80°} - \frac{\sin 20°}{\cos 60° \cos 40°}$

$= \frac{\sin 80° \cos 60° \cos 40° - \sin 20° \cos 20° \cos 80°}{\cos 20° \cos 40° \cos 60° \cos 80°}$

$= \frac{\frac{1}{2}\sin 40°}{\frac{1}{16}} = 8\sin 40°$

60. (5) : If A = 18°, then $5A$ = 90° $\Rightarrow$ $\tan 5A = \infty$

$\Rightarrow \frac{5\tan A - {}^5C_3 \tan^3 A + {}^5C_5 \tan^5 A}{1 - {}^5C_2 \tan^2 A + {}^5C_4 \tan^4 A} = \infty$

$\Rightarrow 5x^4 - 10x^2 + 1 = 0$, where $x = \tan A$

61. (3) : We have $\Sigma \cot A \cot B = 1$

$2\cot B = \cot A + \cot C$

By A.M. ≥ G.M. $\frac{\cot A + \cot C}{2} \geq \sqrt{\cot A \cot C}$

$\Rightarrow \cot^2 B \geq \cot A \cot C$

$\Rightarrow \cot B \geq \sqrt{1 - 2\cot^2 B} \Rightarrow \cot^2 B \geq \frac{1}{3}$

$\Rightarrow \cot B \geq \frac{1}{\sqrt{3}}$

62. (8) : $\frac{\sec^4 \alpha}{\tan^2 \beta} + \frac{\sec^4 \beta}{\tan^2 \alpha}$

Let $\tan^2\alpha = a$, $\tan^2\beta = b$

$\frac{(1+a)^2}{b} + \frac{(1+b)^2}{a} = \frac{a^2+1}{b} + \frac{b^2+1}{a} + 2\left(\frac{a}{b} + \frac{b}{a}\right)$

Since $\frac{\frac{a}{b} + \frac{b}{a}}{2} \geq 1$

$\Rightarrow \frac{\frac{a^2+1}{b} + \frac{b^2+1}{a}}{2} \geq \sqrt{\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)}$

$a + \frac{1}{a} \geq 2 \ ; \ b + \frac{1}{b} \geq 2$

$\therefore \text{Min}\left(\frac{\sec^4\alpha}{\tan^2\beta} + \frac{\sec^4\beta}{\tan^2\alpha}\right) = 4 + 4 = 8$

63. (1) : $\frac{(\sin x)^{2y}}{(\cos x)^{y^2/2}} + \frac{(\cos x)^{2y}}{(\sin x)^{y^2/2}} \geq 2\sqrt{\frac{(\sin x \cos x)^{2y}}{(\cos x \sin x)^{y^2/2}}}$

$= 2(\sin x \cos x)^{y - \frac{y^2}{4}}$

$\Rightarrow \sin 2x \geq 2(\sin x \cos x)^{y - \frac{y^2}{4}} \Rightarrow y - \frac{y^2}{4} \geq 1$

$\Rightarrow y = 2$ and $\sin x = \cos x \ \therefore (x, y) = \left(\frac{\pi}{4}, 2\right)$

64. (2) : If $\theta \in \left(0, \frac{\pi}{2}\right)$ then sinθ and cosθ both are positive $\forall \theta \in \left(0, \frac{\pi}{2}\right)$,

$3\sin A < \sin A + \sin B + \sin C < 3\sin C$

and $\frac{1}{3\cos A} < \frac{1}{\cos A + \cos B + \cos C} < \frac{1}{3\cos C}$

$\Rightarrow \tan A < \frac{\sin A + \sin B + \sin C}{\cos A + \cos B + \cos C} < \tan C$

So maximum value of n is 2.

65. (1) : $(1 + \tan x^\circ)(1 + \tan(x + 1)^\circ) \ldots$

$\ldots(1 + \tan(x + 44)^\circ)$

$\geq (1 + \tan 1^\circ)(1 + \tan 2^\circ) \ldots (1 + \tan 45^\circ)$

Since $(1 + \tan 1^\circ)(1 + \tan 2^\circ)(1 + \tan 3^\circ) \ldots$

$\ldots (1 + \tan 43^\circ)(1 + \tan 44^\circ)(1 + \tan 45^\circ)$

$= (1 + \tan 1^\circ)(1 + \tan 44^\circ)(1 + \tan 2^\circ)(1 + \tan 43^\circ)$

$(1 + \tan 3^\circ)(1 + \tan 42^\circ) \ldots (1 + \tan 22^\circ)(1 + \tan 23^\circ) \times 2$

Using $(1 + \tan x^\circ)(1 + \tan(45 - x)^\circ) = 2$

We have

$(1 + \tan 1^\circ)(1 + \tan 2^\circ)(1 + \tan 3^\circ) \ldots (1 + \tan 44^\circ)$

$(1 + \tan 45^\circ) = 2^{23}$

$\Rightarrow (1 + \tan x^\circ)(1 + \tan(x + 1)^\circ)(1 + \tan(x + 2)^\circ) \ldots$

$(1 + \tan(x + 44)^\circ) \leq 2^{23}$

Equality occurs at $x = 1$

66. (2) : $\left(2\sin\left(x+\frac{\pi}{6}\right)\right)^{\left|2\sin\left(x+\frac{\pi}{6}\right)\right|} = 4$

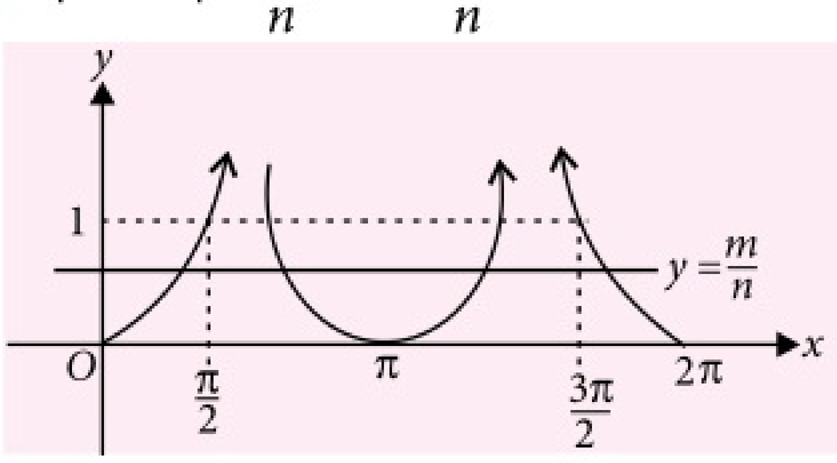
$2\sin\left(x+\frac{\pi}{6}\right) = \pm 2 \Rightarrow \sin\left(x+\frac{\pi}{6}\right) = \pm 1$

$\Rightarrow x+\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{2} \Rightarrow x = \frac{\pi}{3}, -\frac{2\pi}{3}$

67. (1) : $\left|\frac{1+x^2}{2x}\right| \leq 1 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$

But $x = -1$ will not satisfy the equation.

68. (4) : $|\tan x| = \frac{m}{n}; \quad 0 < \frac{m}{n} < 1$



69. (1) : $\tan x > 0$, $\tan x \neq 1$

$$(\log_{\sqrt{3}} \tan x)\left(\sqrt{\frac{\log_{\sqrt{3}}(\sqrt{3})^3}{\log_{\sqrt{3}}\sqrt{3}} + \frac{\log_{\sqrt{3}} 3}{\log_{\sqrt{3}} \tan x}}\right) = -1$$

Let $\log_{\sqrt{3}} \tan x = t < 0$

$\Rightarrow t\sqrt{3+\frac{2}{t}} = -1$

$\Rightarrow t^2\left(3+\frac{2}{t}\right) = 1$

$\Rightarrow 3t^2 + 2t - 1 = 0$

$\Rightarrow t = -1$ and $t = \frac{1}{3}$ (not possible)

$\Rightarrow \log_{\sqrt{3}} \tan x = -1$

$\Rightarrow \tan x = \frac{1}{\sqrt{3}} \Rightarrow x = \frac{\pi}{6}$ only in $[0, \pi]$

70. (0) : It is possible only if $\sin x = \sin 2x = \sin 3x = \sin 4x = 1$ which donot occur simultaneously.

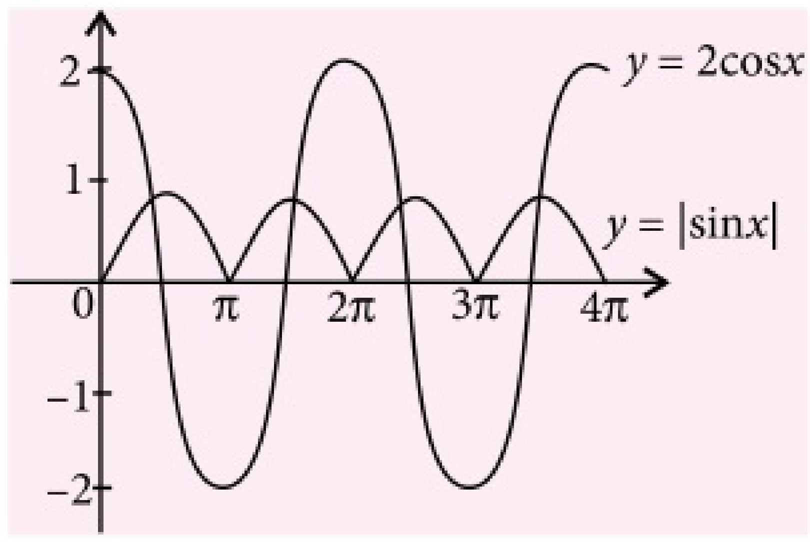
71. (4) : $\sin x[\sin^3 x - \cos^2 x + 2\sin x + 1] = 0$

$\Rightarrow \sin x[\sin^3 x + \sin^2 x + 2\sin x] = 0$

$\Rightarrow \sin^2 x = 0$ or $\sin^2 x + \sin x + 2 = 0$

$\Rightarrow x = 0, \pi, 2\pi, 3\pi$

72. (4) :



concept BOOSTERS

Class XII

Differential Equations and Area Under Curve

* ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

DIFFERENTIAL EQUATIONS

DEFINITION

- An equation that involves independent and dependent variables and the derivatives of the dependent variables is called a **Differential Equation**.
- A differential equation is said to be ordinary, if the differential coefficients have reference to a single independent variable only.
- Finding the unknown function is called **Solving or Integrating** the differential equation. The solution of the differential equation is also called its **Primitive**, because the differential equation can be regarded as a relation derived from it.
- The order of a differential equation is the order of the highest differential coefficient occurring in it.
- The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occurring in it , after it has been expressed in a form free from radicals and fractions so far as derivatives are concerned, thus the differential equation :

$$f(x, y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x, y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q = 0$$

is of order m and degree p.
Note that in the differential equation
$e^{y'''} - xy'' + y = 0$, order is 3 but degree doesn't apply.

FORMATION OF A DIFFERENTIAL EQUATION

If an equation in independent and dependent variables having some arbitrary constants is given , then a differential equation is obtained as follows :

- Differentiate the given equation w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it.
- Eliminate the arbitrary constants .
The eliminant is the required differential equation. Consider forming of a differential equation for $y^2 = 4a(x + b)$, where a and b are arbitrary constants.
- A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

GENERAL AND PARTICULAR SOLUTIONS

- The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **General Solution (or Complete Integral or Complete Primitive).**
A solution obtained from the general solution by giving particular values to the constants is called a **Particular Solution**.
Note that the general solution of a differential equation of the n^{th} order contains 'n' and only 'n' independent arbitrary constants. The arbitrary

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constants in the solution of a differential equation are said to be independent, when it is impossible to deduce from the solution an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = Ae^{x+B}$ are not independent since the equation can be written as $y = Ae^B. e^x = Ce^x$. Similarly, the solution of $y = A \sin x + B \cos (x + C)$ appears to contain three arbitrary constants, but they are really equivalent to two only.

ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS

- **VARIABLE SEPARABLE :** If the differential equation can be expressed as; $f(x)dx + g(y)dy = 0$, then this is said to be variable - separable type. A general solution of this type is given by $\int f(x)dx + \int g(y)dy = c$,

  where c is the arbitrary constant. Consider the example : $(dy/dx) = e^{x-y} + x^2 \cdot e^{-y}$.

  Note : Sometimes transformation to the polar coordinates facilitates separation of variables.

  In this connection, it is convenient to remember the following differentials.

  (a) If $x = r \cos\theta$; $y = r \sin\theta$, then
  (i) $x\,dx + y\,dy = r\,dr$
  (ii) $dx^2 + dy^2 = dr^2 + r^2\,d\theta^2$
  (iii) $x\,dy - y\,dx = r^2\,d\theta$

  (b) If $x = r \sec\theta$; $y = r \tan\theta$, then
  $x\,dx - y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2 \sec\theta\,d\theta$.

  (c) $\dfrac{dy}{dx} = f(ax + by + c), b \neq 0$

  To solve this , substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

  Consider the example $(x+y)^2 \dfrac{dy}{dx} = a^2$

- **HOMOGENEOUS EQUATIONS:** A differential equation of the form $\dfrac{dy}{dx} = \dfrac{f(x, y)}{\phi(x, y)}$

  where $f(x, y)$ and $\phi(x, y)$ are homogeneous functions of x and y, and of the same degree, is called **Homogeneous** differential equation.

  This equation may also be reduced to the form $\dfrac{dy}{dx} = g\left(\dfrac{x}{y}\right)$ and is solved by putting $y = vx$ so that the dependent variable y is changed to another variable v, where v is some unknown function, the differential equation is transformed to an equation with variable separable.

  Consider $\dfrac{dy}{dx} + \dfrac{y(x+y)}{x^2} = 0$.

- **EQUATIONS REDUCIBLE TO THE HOMOGENEOUS FORM :**

  If $\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$; where $a_1b_2 - a_2b_1 \neq 0$

  i.e. $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$,

  then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables u and v, where h and k are arbitrary constants to be chosen so as to make the given equation homogeneous.

  (a) $a_1b_2 - a_2b_1 = 0$, then a substitution $u = a_1x + b_1y$ transforms the differential equation to an equation with variable separable.

  (b) $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\,dy + y\,dx$ and integrating term by term yields the result easily.

  Consider $\dfrac{dy}{dx} = \dfrac{x - 2y + 5}{2x + y - 1}$; $\dfrac{dy}{dx} = \dfrac{2x + 3y - 1}{4x + 6y - 5}$

  and $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{6x - 5y + 4}$

- In an equation of the form $yf(xy)dx + xg(xy)dy = 0$, the variables can be separated by the substitution $xy = v$.

RESULTS

(a) The function $f(x, y)$ is said to be a homogeneous function of degree n if for any real number t ($\neq 0$) , we have $f(tx, ty) = t^n f(x, y)$.
e.g, $f(x , y) = ax^{2/3} + hx^{1/3}.y^{1/3} + by^{2/3}$ is a homogeneous function of degree 2/3.

(b) A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is said to be homogeneous if $f(x, y)$ is a homogeneous function of degree zero *i.e.*, $f(tx, ty) = t^0 f(x, y) = f(x, y)$. The function f does not depend on x and y separately but only on their ratio $\frac{y}{x}$ or $\frac{x}{y}$.

LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if the dependent variable and its differential coefficients occur in the first degree only and are not multiplied together.

The n^{th} order linear differential equation is of the form;

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n(x)\cdot y = \phi(x),$$

where $a_0(x)$, $a_1(x)$,....., $a_n(x)$ are called the coefficients of the differential equation.

Note that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. *e.g.* the differential equation $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER

The most general form of a linear differential equation of first order is $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x.

To solve such an equation multiply both sides by $e^{\int Pdx}$.

REMARKS

- The factor $e^{\int Pdx}$ on multiplying by which the left hand side of the differential equation becomes the differential coefficient of some function of x and y, is called integrating factor of the differential equation popularly abbreviated as I. F.
- It is very important to remember that on multiplying by the integrating factor , the left hand side becomes the derivative of the product of y and the I. F.
- Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. *e.g.* the equation; $(x + y + 1)\frac{dy}{dx} = y^2 + 3$ can be written as $(y^2 + 3)\frac{dx}{dy} = x + y + 1$, which is a linear differential equation.

EQUATIONS REDUCIBLE TO LINEAR FORM

The equation $\frac{dy}{dx} + Py = Q \cdot y^n$ are where P and Q functions of x, is reducible to the linear form by dividing it by y^n and then substituting $y^{-n+1} = Z$. Consider the example $(x^3y^2 + xy)dx = dy$

The equation $\frac{dy}{dx} + Py = Q \cdot y^n$ is called **Bernouli's Equation**.

TRAJECTORIES

Suppose we are given the family of plane curves, $\Phi(x, y, a) = 0$ depending on a single parameter a.

A curve making at each of its points a fixed angle α with the curve of the family passing through that point is called an *isogonal trajectory* of that family; if in particular $\alpha = \pi/2$, then it is called an *orthogonal trajectory*.

Orthogonal trajectories : We set up the differential equation of the given family of curves. Let it will be of the form

$$F(x, y, y') = 0$$

The differential equation of the orthogonal trajectories is of the form

$$F\left(x,\ y,\ -\frac{1}{y'}\right) = 0$$

The general integral of this equation

$$\Phi_1(x, y, C) = 0$$

gives the family of orthogonal trajectories.

REMARKS

- $d\left(-\frac{1}{xy}\right) = \frac{x\,dy + y\,dx}{x^2 y^2}$

- $\dfrac{dx+dy}{x+y} = d(\ln(x+y))$
- $\dfrac{y\,dx - x\,dy}{x^2+y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$
- $xdy + ydx = d(xy)$
- $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
- $\dfrac{y\,dx - x\,dy}{x\,y} = d\left(\ln\dfrac{x}{y}\right)$
- $d\left(\dfrac{e^y}{x}\right) = \dfrac{x\,e^y\,dy - e^y\,dx}{x^2}$
- $d\left(\dfrac{e^x}{y}\right) = \dfrac{y\,e^x\,dx - e^x\,dy}{y^2}$
- $\dfrac{x\,dx + y\,dy}{x^2+y^2} = d\left[\ln\sqrt{x^2+y^2}\right]$
- $\dfrac{x\,dy + y\,dx}{x\,y} = d(\ln xy)$

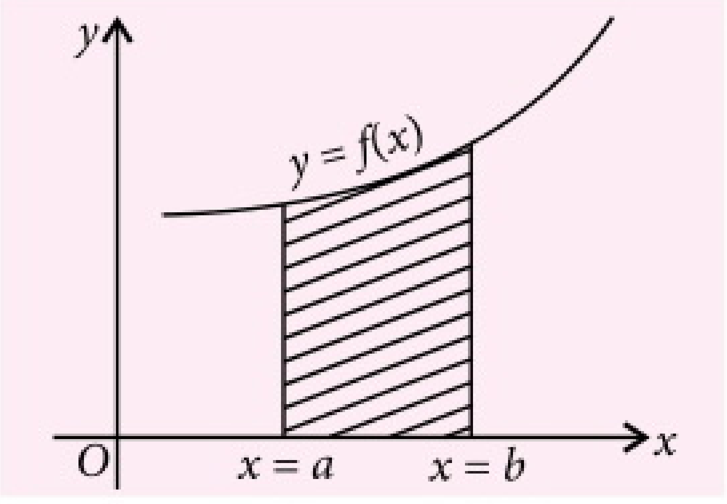
AREA UNDER CURVE

RESULTS

- The area bounded by the curve $y = f(x)$, the x-axis and the ordinates at $x = a$ & $x = b$ is given by,

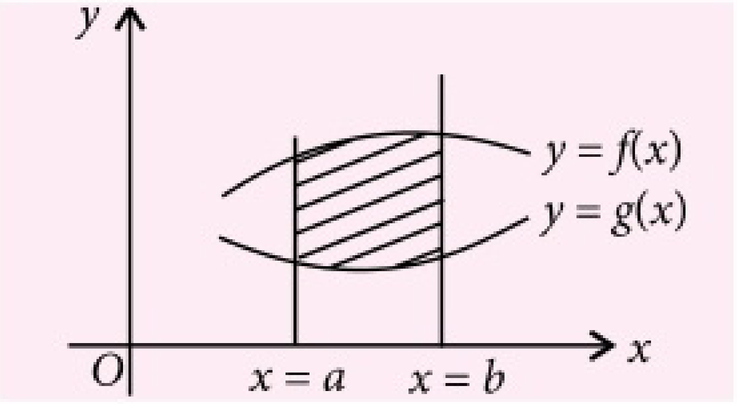
$$A = \int_a^b f(x)dx = \int_a^b y\,dx$$

- If the area is below the x-axis, then A is negative. The convention is to consider the magnitude only *i.e.* $A = \left|\displaystyle\int_a^b y\ dx\right|$ in this case.



- Area between the curves $y = f(x)$, $y = g(x)$ and the ordinates at $x = a$ & $x = b$ is given by,

$$A = \int_a^b f(x)dx - \int_a^b g(x)\,dx = \int_a^b [f(x) - g(x)]dx$$



- Average value of a function $y = f(x)$ w.r.t. x over an interval $a \le x \le b$ is defined as :

$$y(av) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

- The area function A_a^x satisfies the differential equation $\dfrac{d\,A_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$.

  If $F(x)$ is any integral of $f(x)$ then,

$$A_a^x = \int f(x)\,dx = F(x) + c$$

$$A_a^a = 0 = F(a) + c \;\Rightarrow\; c = -F(a)$$

  Hence $A_a^x = F(x) - F(a)$. Finally by taking $x = b$, we get, $A_a^b = F(b) - F(a)$.

CURVE TRACING

The following outline procedure is to be applied in sketching the graph of a function $y = f(x)$ which in turn will be extremely useful to quickly and correctly evaluate the area under the curves.

Symmetry : The symmetry of the curve is judged as follows :

- If on interchanging the signs of x & y both, the equation of the curve is unaltered, then there is symmetry in opposite quadrants.
- If the equation of the curve remains unchanged on interchanging x and y, then the curve is symmetrical about $y = x$.
- If all the powers of y in the equation are even, then the curve is symmetrical about the x-axis.
- Find dy/dx & equate it to zero to find the points on the curve where you have horizontal tangents.

- Find the points where the curve crosses the x-axis & also the y-axis.
- Examine if possible the intervals when $f(x)$ is increasing or decreasing. Examine what happens to 'y' when $x \to \infty$ or $-\infty$.
- If all the powers of x are even, the curve is symmetrical about the axis of y.
- If powers of x & y both are even, the curve is symmetrical about the axis of x as well as of y.

REMARKABLE RESULTS

- Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $8a^2/3m^3$.
- Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4by$ is $16ab/3$.
- Whole area of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ is πab.

PROBLEMS
SECTION-I

Single Correct Answer Type

1. A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity of water in reservoir A is 1.5 times the quantity of water in reservoir B. After how many hours do both the reservoirs have the same quantity of water?

(a) 2 (b) $\log_e 2$
(c) $\log_{4/3} 2$ (d) $\log_{4/3} 4$

2. If a and b are arbitrary constants, then the differential equation whose solution is $(x - a)^2 + (y - b)^2 = r^2$ is

(a) $\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = r^2 \frac{d^2y}{dx^2}$

(b) $\left(1+\frac{dy}{dx}\right)^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$

(c) a second order and third degree differential equation

(d) $\left(1+\left(\frac{dy}{dx}\right)^2\right)^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$

3. The general solution of the differential equation $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0$ is $2xe^{f(y)}=e^{2f(y)}+c$, then the area of the region bounded by the curves $x=f(y)$, $y=\pm\sqrt{3}$ and y-axis is

(a) $\frac{\pi}{\sqrt{3}}-\log 2$ (b) $\frac{2\pi}{\sqrt{3}}-\log 4$

(c) $\frac{\pi}{\sqrt{3}}+\log 2$ (d) $\frac{2\pi}{\sqrt{3}}$

4. If $\frac{dy}{dx}+\frac{y}{x}=x^2$, then $2y(2)-y(1)=$

(a) $\frac{9}{4}$ (b) $\frac{11}{4}$ (c) $\frac{13}{4}$ (d) $\frac{15}{4}$

5. The solution of $e^{xy}(xy^2dy + y^3dx) + e^{x/y}(ydx - xdy) = 0$ is

(a) $e^{xy} - e^{x/y} + c = 0$ (b) $e^{xy} - e^{y/x} + c = 0$
(c) $e^{xy} + e^{x/y} + c = 0$ (d) None of these

6. The solution of the differential equation $\frac{dy}{dx}=\frac{x-2y+3}{2x-y+5}$ is $f(x, y, c) = 0$. For suitable values of c, $f(x, y, c) = 0$ represents a pair of lines whose point of intersection is

(a) $\left(-\frac{7}{3},\frac{2}{3}\right)$ (b) $\left(-\frac{7}{3},\frac{4}{3}\right)$

(c) $\left(-\frac{7}{3},\frac{1}{3}\right)$ (d) $\left(\frac{7}{3},-\frac{1}{3}\right)$

7. If the differential equation of a curve, passing through $\left(0,-\frac{\pi}{4}\right)$ and $(t, 0)$ is

$\cos y\left(\frac{dy}{dx}+e^{-x}\right)+\sin y\left(e^{-x}-\frac{dy}{dx}\right)=e^{e^{-x}}$, then

the value of $t\cdot e^{e^{-t}}$ is

(a) -1 (b) 1 (c) 2 (d) -2

8. Solution of $\frac{dy}{dx}=\frac{y^3}{e^{2x}+y^2}$ is

(a) $e^{-2x}y^2+2\ln|y|=c$

(b) $e^{2x}y^2-2\ln|y|=c$

(c) $e^x+\ln|y|=c$ (d) none of these

9. The primitive of the differential equation $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$ is

(a) $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$

(b) $x^2e^y - \dfrac{x^2}{y} + \dfrac{x}{y^3} = c$

(c) $x^2e^y + \dfrac{x^2}{y} - \dfrac{x}{y^3} = c$

(d) $x^2e^y - \dfrac{x^2}{y} - \dfrac{x}{y^3} = c$

10. The differential equation $e^y dx + (e^y \cdot x + 2y)\,dy = 0$ has the particular solution $y(0) = 1$. The value of x, when $y = 0$, is

(a) 1 (b) 0 (c) 2 (d) -1

11. If the independent variable x is changed to y, then the differential equation $x\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^3-\dfrac{dy}{dx}=0$ is transformed to $x\dfrac{d^2x}{dy^2}+\left(\dfrac{dx}{dy}\right)^2=k$, k being a number. Then k equals

(a) 0 (b) 1 (c) -1 (d) 2

12. If the differential equation $\dfrac{dx}{3y+f}+\dfrac{dy}{px+g}=0$ represents a circle, then p =_______

(a) g (b) f (c) 4 (d) 3

13. The solution of the differential equation $2x^3ydy + (1 - y^2)(x^2y^2 + y^2 - 1)dx = 0$ is [where c is a constant]

(a) $x^2y^2 = (cx + 1)(1 - y^2)$

(b) $x^2y^2 = (cx + 1)(1 + y^2)$

(c) $x^2y^2 = (cx - 1)(1 - y^2)$

(d) none of these

14. A normal at any point (x, y) to the curve $y = f(x)$ cuts triangle of unit area with the axes, the equation of the curve is

(a) $y^2 - x^2\left(\dfrac{dy}{dx}\right)^2 = 4\dfrac{dy}{dx}$

(b) $x^2 - y^2\left(\dfrac{dy}{dx}\right)^2 = \dfrac{dy}{dx}$

(c) $x + y\dfrac{dy}{dx} = y$

(d) $x^2 + 2xy\dfrac{dy}{dx} + y^2\left(\dfrac{dy}{dx}\right)^2 = 2\dfrac{dy}{dx}$

15. The curve $x + y - \ln(x + y) = 2x + 5$ has a horizontal normal at the point (α, β). Then $\alpha + \beta$ is equal to

(a) -1 (b) 1 (c) 2 (d) -2

SECTION-II

More than One Correct Answer Type

16. The graph of the function $y = f(x)$ passing through the point (0, 1) and satisfying the differential equation $\dfrac{dy}{dx} + y\cos x = \cos x$ is such that

(a) it is a constant function
(b) it is periodic
(c) it is neither an even nor an odd function
(d) it is continuous & differentiable for all x

17. For the differential equation

$y=\ln\left\{1+y'+\dfrac{(y')^2}{2!}+\dfrac{(y')^3}{3!}+......\infty\right\}$, where $y'=\dfrac{dy}{dx}$

(a) order is 1
(b) degree is 1
(c) order is not defined
(d) degree is not defined

18. Consider the family of curves $C : y = cx^2$.

(a) The differential equation of C is $xdy - 2ydx = 0$
(b) The differential equation of orthogonal trajectory of C is $x\,dx + 2ydy = 0$
(c) The equation of family of orthogonal trajectories of C is $x^2 + 2y^2 = c$
(d) The equation of family of orthogonal trajectory of C is $2x^2 + y^2 = c$

19. The solution of $\dfrac{dy}{dx}=\dfrac{ax+h}{by+k}$ represents a parabola if

(a) $a = -2, b = 0$ (b) $a = -2, b = 2$
(c) $a = 0, b = 2$ (d) $a = 0, b = 0$

20. $\left(\dfrac{dy}{dx}\right)^2 + 2y\cot x\dfrac{dy}{dx} = y^2$ has the solution

(a) $y-\frac{c}{1+\cos x}=0$ (b) $y=\frac{c}{1-\cos x}$

(c) $x=2\sin^{-1}\sqrt{\frac{c}{2y}}$ (d) $x=2\cos^{-1}\left(\frac{c}{2y}\right)$

21. The normal at a general point (a, b) on curve makes an angle θ with x-axis which satisfies $b(a^2\tan\theta - \cot\theta) = a(b^2 - 1)$. The equation of curve can be

(a) $y = e^{x^2/2} + c$ (b) $\log(ky^2) = x^2$

(c) $y = ke^{x^2/2}$ (d) $x^2 + y^2 = k$

22. The differential equation $\frac{dx}{dy}=\frac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity

(a) $\sqrt{\frac{3}{5}}$ (b) $\sqrt{\frac{5}{3}}$ (c) $\sqrt{\frac{2}{5}}$ (d) $\sqrt{\frac{5}{2}}$

23. Solution of the differential equation

$xy\left(\frac{dy}{dx}\right)^2-(x^2-y^2).\frac{dy}{dx}-xy=0$ is/are

(a) A hyperbola whose eccentricity is $\sqrt{2}$

(b) $xy = c$

(c) $(xy - c)\cdot(y^2 - x^2 - c) = 0$

(d) $(x^2 + y^2 - c)\cdot(x + y - c) = 0$

24. The solution of $x^2y_1^2 + xyy_1 - 6y^2 = 0$ is/are

(a) $y = Cx^2$ (b) $x^2y = C$

(c) $\frac{1}{2}\log y=C+\log x$ (d) $x^3y = C$

SECTION-III

Comprehension Type

Paragraph for Question No. 25 and 26

For certain curve $y=f(x)$ satisfying $\frac{d^2y}{dx^2}=6x-4$, $f(x)$ has local minimum value 5 when $x = 1$. Answer the following questions.

25. Number of critical points for $y = f(x)$ for $x \in (0, 1)$ are

(a) 0 (b) 1

(c) 2 (d) 3

26. Global maximum value of $y = f(x)$ for $x \in [0, 2]$ is

(a) 5 (b) 7

(c) 8 (d) 9

Paragraph for Question No. 27 to 29

An equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$, is called a linear differential equation of the first order. Its solution is given as $ye^{\int Pdx}=\int Qe^{\int Pdx}+k$, k being the constant of integration. A number of differential equations can be reduced to the linear differential equation and then one can solve them.

27. The general solution of the differential equation $\frac{dy}{dx}=e^{x-y}(e^x-e^y)$ is, (c being the constant of integration)

(a) $e^y=e^x-1+ce^{-e^x}$ (b) $e^y=1-e^x+ce^{-e^x}$

(c) $e^x=e^y-1+ce^{-e^x}$ (d) $e^x=1-e^y+ce^{-e^x}$

28. The general solution of the differential equation $\frac{dy}{dx}+\frac{y\ln y}{x}=\frac{y}{x^2}(\ln y)^2$ is, (c being the constant of integration)

(a) $2x = (1 - cx^2)\ln y$ (b) $x = (1 + 2cx^2)\ln y$

(c) $2x = (1 + 2cx^2)\ln y$ (d) $x = (1 + cx^2)\ln y$

29. The general solution of the differential equation $(y^2 + e^{2x})dy - y^3dx = 0$, ($c$ being the constant of integration), is

(a) $y^2e^{-2x} + 2\ln y = c$ (b) $y^2e^{-2x} - 2\ln y = c$

(c) $y^2e^{-x} - 2\ln y = c$ (d) $y^2e^{-x} + 2\ln y = c$

Paragraph for Question No. 30 to 32

Consider the differential equation

$xyp^2-(x^2-y^2)p-xy=0$, where $p=\frac{dy}{dx}$

30. A solution of the above differential equation passing through (2 , 2) is

(a) $x^2 + y^2 = 8$ (b) $y^2 = 2x$

(c) $xy = 4$ (d) $x^2y = 1$

31. Another solution of above differential equation satisfying the point(2,3) is

(a) $xy = 5$ (b) $x^2 + y^2 = 13$

(c) $y^2 - x^2 = 5$ (d) $y^2 = 4x + 1$

32. The equations of orthogonal trajectories of the solutions of the given differential equation are

(a) $xy = c_1$, $x^2 - y^2 = c_2$

(b) $xy = c_1$, $x^2 + y^2 = c_2$

(c) $x^2 - y^2 = c_1$, $x^2 + y^2 = c_2$

(d) None of these

SECTION-IV

Matrix Match Type

33. Match the following :

	Column-I		Column-II
(A)	The function y defined by the equation $xy - \log y = 1$ satisfies $x(yy'' + y'^2) - y'' + kyy' = 0$. The value of k is	(p)	3
(B)	If the function $y(x)$ represented by $x = \sin t,\ y = ae^{t\sqrt{2}} + be^{t\sqrt{2}}$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies the equation $(1 - x^2)y'' - xy' = ky$, then $k + 1$ is equal to	(q)	2
(C)	Let $F(x) = f(x)g(x)\,h(x)$ for all real x, where $f(x)$, $g(x)$ and $h(x)$ are differentiable function. At some point x_0, if $F'(x_0) = 21\,F(x_0)$, $f'(x_0) = 4\,f(x_0)$, $g'(x_0) = -7g(x_0)$ and $h'(x_0) = kh(x_0)$, then $\frac{k}{8}$ is equal to	(r)	1
(D)	Let $f(x) = x^n$, n being a non-negative integer. The number of values of n for which the equality $f'(a + b) = f'(a) + f'(b)$ is valid for all $a, b > 0$ is	(s)	4

34. The differential equation for which the solution is

	Column-I		Column-II
(A)	$y = ae^{3x} + be^{5x}$	(p)	$y_2 + 16y = 0$
(B)	$xy = ae^x + be^{-x} + x^2$	(q)	$y_2 - 8y_1 + 15y = 0$
(C)	$ax^2 + by^2 = 1$	(r)	$xy_2 + 2y_1 - xy + x^2 - 2 = 0$
(D)	$y = a\sin(4x + b)$	(s)	$x(yy_2 + y_1^2) = yy_1$

35. Match the following

	Column-I		Column-II
(A)	Solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}(e^x + e^{-x}) + 1 = 0$ is given by	(p)	$x + y + 2 = ce^y$, (where c is arbitrary) constant)
(B)	Solutions of the differential equation $(x + y + 1)dy = dx$ are given by	(q)	$y = e^x + cx$ (where c is arbitrary constant)
(C)	Solution of the differential equation $(x + y + 2)dx + (2x + 2y - 1)dy = 0$ are	(r)	$y + e^{-x} = c$ (where c is arbitrary constant)
(D)	Solution of $x\frac{dy}{dx} - y = (x-1)e^x$ is	(s)	$2(x + y + 2) + 5\ln(x + y - 3) = x + c$, (where c is arbitrary constant)

SECTION-V

Integer Answer Type

36. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ and $Kx\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$, then the value of constant K must be

37. If $a + b + c = 0$ and

$$y = \frac{1}{x^b + x^{-c} + 1} + \frac{1}{x^c + x^{-a} + 1} + \frac{1}{x^a + x^{-b} + 1},$$

then $\frac{dy}{dx}$ is equal to

38. The order and degree of the differential equation satisfying $\sqrt{1-x^4} + \sqrt{1-y^4} = a(x^2 - y^2)$ are P and Q, then $P + Q$ is (a is arbitrary constant)

39. The equation of a curve whose slope at any point is thrice its abscissa and which passes through (–1, –3) is $2y = \lambda(x^2 - 3)$, then the value of λ must be = ______

40. If $y = y(x)$ and $\frac{2+\sin x}{y+1}\frac{dy}{dx} = -\cos x,\ y(0) = 1$, then $y\left(\frac{\pi}{2}\right) + \frac{8}{3} =$

41. The order of differential equation whose general solution is given by

$y = C_1\cos(x + C_2) - C_3e^{C_4 - x} + C_5\sin x$, where C_1, C_2, C_3, C_4, C_5 are arbitrary constants is

42. If $y = C_1e^{2x} + C_2e^x + C_3e^{-x}$ satisfies the differential equation $\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ then $a^2 + b^2 + c^2 =$

43. A curve is such that the intercept of its tangent on the x-axis is twice the abscissa and which passes through the point (1 , 2). If the ordinate of the point on the curve is $\frac{1}{3}$, then the value of the abscissa is

44. The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$ and the curve passes through the point (1, 2). If the area of the region bounded by the curve , the x-axis and the line $x = 1$ in first quadrant is A then $6A$ =

45. The order of the differential equation of the family of all circles with one diameter along the line $2x + 3y - 5 = 0$ is

46. If solution of the differential equation

$$y(2x^4 + y)\frac{dy}{dx} = (1 - 4xy^2)x^2 \text{ is given by}$$

$\lambda_1 x^4 y^2 + \lambda_2 x^3 + \lambda_3 y^3 = c$, where c is an arbitrary constant, $\lambda_1, \lambda_2, \lambda_3$ are relatively prime integers then $\lambda_1 + \lambda_2 + \lambda_3 =$ ______

47. For any differentiable function $y = f(x)$, the value of $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 \frac{d^2x}{dy^2}$ is

48. The solution of the differential equation

$$x = 1 + xy\frac{dy}{dx} + \frac{x^2y^2}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{x^3y^3}{3!}\left(\frac{dy}{dx}\right)^3 + \ldots\ldots \text{ is}$$

$y^k = (\ln x)^k + c$. Then $k =$ ______

49. If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\dfrac{\frac{d^3y}{dx^3} - 13\frac{dy}{dx}}{y} = 4K$, then the value of K is

50. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$ is

51. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$ is $\frac{1}{x^5y^5} = \frac{p}{qx^2} + c$, then the value of $p + q$ is ______

52. The orthogonal trajectory of the family of rectangular hyperbola $xy = c^2$ is $ax^2 + by^2 = k$, then the value of $a + b$ is ______

53. The order of the differential equation obtained by eliminating arbitrary constants from $y = c_1 e^{x + c_2} + c_3 e^x$ is ______

54. The degree of the differential equation of all circles in the first quadrant which touch the coordinate axes is ______

55. A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x and y axes at A and B respectively such that $AP : PB = 1 : 3$. If $f(1) = 1$, then the curve passes through $\left(k, \frac{1}{8}\right)$ where $k =$ ______

56. Slope of the tangent at any point (x, y) on the curve $y = f(x)$ is $\frac{y}{x^2}$ and the curve passes through the point (1, 3), then $\underset{x \to \infty}{\text{Lt}} f(x) = Ke$, where $K =$ ______

57. If the solution of $\frac{x}{x^2 + y^2}dy = \left(\frac{y}{x^2 + y^2} - 1\right)dx$ satisfies $y(0) = 1$, then the value of $\frac{16}{\pi} y\left(\frac{\pi}{4}\right)$ is

SOLUTIONS

1. (c) : $\frac{du}{dt} = -k_1 u$

$\Rightarrow \frac{du}{u} = -k_1 dt \Rightarrow \log u = -k_1 t + c$

When $t = 0$, $u = u_0$ $\therefore$ $c = \log u_0$

$\log u = -k_1 t + \log u_0$

$\Rightarrow \log\left(\frac{u}{u_0}\right) = -k_1 t \Rightarrow u = u_0 e^{-k_1 t}$

Similarly $V = V_0 e^{-k_2 t}$

Given that $u_0 = 2V_0$ at $t = 0$

At $t = 1$ hour, $u = \frac{3}{2}V$

$\therefore$ $u_0 e^{-k_1} = \frac{3}{2}V_0 e^{-k_2} \Rightarrow 2V_0 e^{-k_1} = \frac{3}{2}V_0 e^{-k_2}$

$\Rightarrow e^{k_1 - k_2} = \frac{4}{3} \Rightarrow k_1 - k_2 = \log\left(\frac{4}{3}\right)$

After t sec, $u = V$

$\Rightarrow u_0 e^{-k_1 t} = V_0 e^{-k_2 t} \Rightarrow 2V_0 e^{-k_1 t} = V_0 e^{-k_2 t}$

$$\Rightarrow \quad e^{(k_1-k_2)t} = 2 \Rightarrow t = \log_{\frac{4}{3}} 2$$

2. (d) : $(x - a)^2 + (y - b)^2 = r^2$

$$\Rightarrow \quad 2(x-a)+2(y-b)\frac{dy}{dx}=0 \quad \Rightarrow \quad \frac{dy}{dx}=-\left(\frac{x-a}{y-b}\right)$$

$$\Rightarrow \quad 1+(y-b)y_2+y_1^2=0 \Rightarrow \quad y_2=-\frac{r^2}{(y-b)^3}$$

$$\Rightarrow \quad 1+y_1^2=-(y-b)y_2 \quad \Rightarrow \quad -(y-b)^3=\frac{r^2}{y_2}$$

$$\Rightarrow \quad (1+y_1^2)^3=-(y-b)^3 y_2^3 \quad \Rightarrow \quad (1+y_1^2)^3=r^2 y_2^2$$

3. (b) : $\dfrac{dx}{dy}+\dfrac{x}{1+y^2}=\dfrac{e^{\tan^{-1}y}}{1+y^2}$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$\therefore$ General solution is $x\cdot e^{\tan^{-1}y}=\displaystyle\int\frac{\left(e^{\tan^{-1}y}\right)^2}{1+y^2}dy$

$$\Rightarrow \quad x.e^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+c$$

$$\Rightarrow \quad 2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+c \quad \therefore \quad f(y)=\tan^{-1}y$$

$$\therefore \quad \text{Area} = \int_{-\sqrt{3}}^{\sqrt{3}} |\tan^{-1}y|\,dy = 2\int_0^{\sqrt{3}} \tan^{-1} y\,dy$$

$$=2\left[(y\tan^{-1}y)_0^{\sqrt{3}}-\int_0^{\sqrt{3}}\frac{y}{1+y^2}dy\right]$$

$$=2\frac{\pi}{\sqrt{3}}-\left[\log(1+y^2)\right]_0^{\sqrt{3}}=\frac{2\pi}{\sqrt{3}}-\log 4$$

4. (d) : We have $\dfrac{dy}{dx}+\dfrac{y}{x}=x^2$

$$\text{I.F.} = \exp\int\frac{dx}{x}=x$$

$\therefore$ Solution is $xy=\displaystyle\int x^2 x\,dx=\frac{x^4}{4}+c$

Put $x=2 \Rightarrow 2y(2)=\dfrac{16}{4}+c$

$$x=1\Rightarrow y(1)=\frac{1}{4}+c$$

$$\therefore \quad 2y(2)-y(1)=\frac{15}{4}$$

5. (c) : $e^{xy}(xdy+ydx)+e^{x/y}\left(\dfrac{ydx-xdy}{y^2}\right)=0$

$$\Rightarrow \quad d(e^{xy})+d(e^{x/y})=0$$

6. (c) : $(x - 2y + 3)dx = (2x - y + 5)dy$

$\Rightarrow$ $2(xdy + ydx) - ydy - xdx + 5dy - 3dx = 0$

$\Rightarrow$ $x^2 - 4xy + y^2 + 6x - 10y + c = 0$

whose point of intersection is $\left(-\dfrac{7}{3},\dfrac{1}{3}\right)$

7. (b) :

$$(\cos y-\sin y)\frac{dy}{dx}+(\cos y+\sin y)e^{-x}=e^{e^{-x}}$$

Put $\cos y + \sin y = u$

$$\Rightarrow \quad \frac{du}{dx}+e^{-x}u=e^{e^{-x}} \quad \Rightarrow \quad ue^{-e^{-x}}=x$$

$$\Rightarrow \quad e^{-e^{-t}}=t \quad (\text{putting } (t,0)) \Rightarrow \quad t\cdot e^{e^{-t}}=1$$

8. (a) : Dividing by e^{2x}, we get

$$\frac{dy}{dx}=\frac{y^3e^{-2x}}{1+y^2e^{-2x}}$$

$$\Rightarrow \quad dy+y^2e^{-2x}dy=y^3e^{-2x}dx$$

$$\Rightarrow \quad \int 2\frac{dy}{y}+\int 2(ye^{-2x}dy-y^2e^{-2x}dx)=0$$

$$\Rightarrow \quad \int 2\frac{dy}{y}+\int d(e^{-2x}y^2)=c \Rightarrow 2\ln|y|+e^{-2x}y^2=c$$

9. (a) : $y^4[2xe^y dx + x^2e^y dy] + y^2[2xydx - x^2dy]$
$+ [ydx - 3xdy] = 0$

$$\Rightarrow d(x^2e^y)+\frac{2xydx-x^2dy}{y^2}+\frac{ydx-3xdy}{y^4}=0$$

$$\Rightarrow \quad d(x^2e^y)+d\left(\frac{x^2}{y}\right)+d\left(\frac{x}{y^3}\right)=0$$

$$\Rightarrow \quad x^2e^y+\frac{x^2}{y}+\frac{x}{y^3}=c$$

10. (a) : The equation is exact. Its general solution is $xe^y + y^2 = 1$.

Use $y(0) = 1$ to get $x = 1$.

11. (b) : $\frac{dy}{dx}=\frac{1}{dx/dy},\frac{d^2y}{dx^2}=-\frac{-1}{(dx/dy)^3}\frac{d^2x}{dy^2}$

then $x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-\left(\frac{dy}{dx}\right)=0$ changes to

$$x\frac{d^2x}{dy^2}+\left(\frac{dx}{dy}\right)^2-1=0$$

12. (d) : $(px+g)dx+(3y+f)dy=0$

Integrating, we get

$$\frac{px^2}{2}+gx+\frac{3y^2}{2}+fy+c=0$$

$\Rightarrow\ p=3$ (for a circle)

13. (c) : $\frac{2y}{(1-y^2)^2}\cdot\frac{dy}{dx}+\frac{y^2}{1-y^2}\frac{1}{x}=\frac{1}{x^3}$

Put $\frac{y^2}{1-y^2}=t \Rightarrow \frac{2y}{(1-y^2)^2}\frac{dy}{dx}=\frac{dt}{dx}$

$$\Rightarrow \frac{dt}{dx}+\frac{t}{x}=\frac{1}{x^3} \Rightarrow t\cdot x=\int\frac{1}{x^2}dx+c$$

$$\Rightarrow x^2y^2=(cx-1)(1-y^2)$$

14. (d) : The equation of the normal at any point $P(x, y)$ to the curve $y=f(x)$ is

$$y-Y=\frac{-1}{\frac{dy}{dx}}(x-X) \Rightarrow \frac{X}{x+y\frac{dy}{dx}}+\frac{Y}{y+\frac{x}{\frac{dy}{dx}}}=1$$

Given that $\frac{1}{2}\left(x+y\frac{dy}{dx}\right)\left(y+\frac{x}{\frac{dy}{dx}}\right)=1$

$$\Rightarrow x^2+2xy\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2=2\frac{dy}{dx}$$

is the required equation.

15. (b) : Given equation of the curve is

$x+y-\ln(x+y)=2x+5$

$$\Rightarrow 1+\frac{dy}{dx}-\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)=2$$

$$\Rightarrow \frac{dy}{dx}=\left(\frac{x+y+1}{x+y-1}\right)_{(\alpha,\beta)} \therefore \frac{dx}{dy}=0 \ \therefore\ \alpha+\beta=1$$

16. (a, b, d) : I.F. $=e^{\int\cos x\,dx}=e^{\sin x}$

$\Rightarrow\ y.e^{\sin x}=\int e^{\sin x}\cos x\,dx=e^{\sin x}+c$

$\Rightarrow\ y=1$

17. (a, b) :

$$y=\ln\left\{1+y'+\frac{(y')^2}{2!}+\frac{(y')^3}{3!}+\dots\infty\right\}=\ln e^{y'}=y'$$

18. (a, b, c) : $\frac{dy}{dx}=2cx$

$$\therefore\ y=cx^2\Rightarrow y=\left(\frac{1}{2x}\frac{dy}{dx}\right)x^2\Rightarrow y=\frac{x}{2}\frac{dy}{dx} \quad ...(1)$$

For orthogonal trajectory, replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ then (1) becomes

$$y=\frac{x}{2}\left(\frac{-dx}{dy}\right) \Rightarrow 2y\,dy+x\,dx=0$$

On solving we get, $2x^2+2y^2=c$

19. (a, c) : By variable separable method, we have

$$\int(by+k)dy=\int(ax+h)dx$$

$$\Rightarrow \frac{by^2}{2}+ky=\frac{ax^2}{2}+hx+c$$

Clearly for $a=-2$, $b=0$ and for $a=0$, $b=2$, it represents a parabola.

20. (a, b) : $\left(\frac{dy}{dx}\right)^2+2y\cot x\frac{dy}{dx}-y^2=0$

$$\Rightarrow \frac{dy}{dx}=\frac{-2y\cot x\pm\sqrt{4y^2\cot^2x+4y^2}}{2}$$

$$\Rightarrow \frac{dy}{dx}=y(-\cot x\pm\operatorname{cosec}x)$$

$$\Rightarrow \frac{dy}{y}=(-\cot x\pm\operatorname{cosec}x)dx$$

$\Rightarrow \log y=[-\log\sin x\pm\log(\operatorname{cosec}x-\cot x)]+\log c$

$$=\log\frac{c(1\pm\cos x)}{\sin^2x}=\log\left(\frac{c}{1\pm\cos x}\right)$$

$$\Rightarrow y=\frac{c}{1\pm\cos x}$$

21. (b, c, d) : Slope of normal $\tan\theta=-\frac{dx}{dy}$

$\therefore$ The given equation at a general point (x, y) becomes

$$y\left(x^2\left(-\frac{dx}{dy}\right)+\frac{dy}{dx}\right)=x(y^2-1)$$

$$\Rightarrow -yx^2+y\left(\frac{dy}{dx}\right)^2=\frac{dy}{dx}\cdot x(y^2-1)$$

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2-x(y^2-1)\frac{dy}{dx}-yx^2=0$$

$\Rightarrow yy'^2 - xy^2y' + xy' - yx^2 = 0$

$\Rightarrow yy'(y' - xy) + x(y' - xy) = 0$

$$\Rightarrow \frac{dy}{dx}=xy \quad\text{or}\quad \frac{dy}{dx}=\frac{-x}{y}$$

$$\Rightarrow \log y=\frac{x^2}{2}+c \text{ or } x^2+y^2=c$$

$$\Rightarrow y=ke^{\frac{x^2}{2}} \Rightarrow \log y^2=x^2-\log k \Rightarrow \log ky^2=x^2$$

22. (b, d) : $\dfrac{dx}{dy}=\dfrac{3y}{2x} \Rightarrow \int 2x\,dx=\int 3y\,dy$

$$\Rightarrow x^2=\frac{3y^2}{2}+c \quad\text{or}\quad \frac{x^2}{3}-\frac{y^2}{2}=\frac{c}{3}$$

If c is positive, then $e=\sqrt{1+\dfrac{2}{3}}=\sqrt{\dfrac{5}{3}}$

If c is negative, then $e=\sqrt{1+\dfrac{3}{2}}=\sqrt{\dfrac{5}{2}}$

23. (a, b, c) :

$$\frac{dy}{dx}=\frac{(x^2-y^2)\pm\sqrt{(x^2-y^2)^2+4x^2y^2}}{2xy}$$

$$=\frac{(x^2-y^2)\pm(x^2+y^2)}{2xy}=\frac{x}{y} \text{ or } \frac{-y}{x}$$

$\Rightarrow xdx = ydy$ or $xdy + ydx = 0$

$\Rightarrow x^2 - y^2 = c$ or $d(xy) = 0 \Rightarrow xy - c = 0$

Hence solutions are $(xy - c) = 0$ or $y^2 - x^2 - c = 0$

24. (a, c, d) : Solving for $\dfrac{dy}{dx}$, we get

$$\left(x\frac{dy}{dx}+3y\right)\left(x\frac{dy}{dx}-2y\right)=0$$

$$\Rightarrow \text{either } x\frac{dy}{dx}+3y=0 \Rightarrow \frac{dy}{y}=\frac{-3dx}{x} \Rightarrow yx$$

or $x\dfrac{dy}{dx}-2y=0 \Rightarrow y=Cx^2$ or $\dfrac{1}{2}\log y=C+\log x$

(25 - 26) : 25. (b) 26. (b)

$$\frac{d^2y}{dx^2}=6x-4 \Rightarrow \frac{dy}{dx}=3x^2-4x+c$$

local min. at $x=1 \Rightarrow \left(\dfrac{dy}{dx}\right)_{x=1}=0 \Rightarrow c=1$

$$\therefore \frac{dy}{dx}=3x^2-4x+1 \Rightarrow y=x^3-2x^2+x+d$$

$x = 1, y = 5 \Rightarrow d = 5 \quad \therefore f(x) = x^3 - 2x^2 + x + 5$

$$f'(x)=0 \Rightarrow x=\frac{1}{3},1$$

Global max. of $f(x) = 7$

27. (a) : $\dfrac{dy}{dx}=e^{x-y}(e^x-e^y)$

$$e^y\frac{dy}{dx}=e^{2x}-e^{x+y} \Rightarrow e^y\frac{dy}{dx}+e^x.e^y=e^{2x}$$

Let $e^y=u$, so that $e^y\dfrac{dy}{dx}=\dfrac{du}{dx}$

We have, $\dfrac{du}{dx}+e^x\cdot u=e^{2x}$

I.F. $=e^{\int e^x dx}=e^{e^x}$

$\therefore$ The solution is $u\cdot e^{e^x}=\int e^{e^x}.e^{2x}dx$

$=\int e^t\cdot t\,dt$, where $t=e^x$

$= e^t(t - 1) + c$

$=e^{e^x}(e^x-1)+c$

Then $u=e^x-1+ce^{-e^x} \Rightarrow e^y=e^x-1+c\,e^{-e^x}$

28. (c) : $\dfrac{dy}{dx}+\dfrac{y\ln y}{x}=\dfrac{y}{x^2}(\ln y)^2$

We have, $\dfrac{1}{y(\ln y)^2}\dfrac{dy}{dx}+\dfrac{1}{x}.\dfrac{1}{\ln y}=\dfrac{1}{x^2}$

Let $\dfrac{-1}{\ln y}=u$, so that $\dfrac{du}{dx}-\dfrac{u}{x}=\dfrac{1}{x^2}$

I.F. $=e^{-\int\frac{dx}{x}}=e^{-\ln x}=1/x$

Then the solution is $u.\frac{1}{x}=\int\frac{1}{x^3}dx+c$

which on simplification gives $2x = [1 + 2cx^2]\ln y$

29. (a) : $\frac{dx}{dy}=\frac{y^2+e^{2x}}{y^3}$

$$\Rightarrow \frac{dx}{dy}=\frac{1}{y}+\frac{e^{2x}}{y^3} \Rightarrow e^{-2x}\frac{dx}{dy}-e^{-2x}\frac{1}{y}=\frac{1}{y^3}$$

Put $\frac{-e^{-2x}}{2}=u$, so that $e^{-2x}\frac{dx}{dy}=\frac{du}{dy}$

$$\Rightarrow \frac{du}{dy}+\frac{2}{y}u=\frac{1}{y^3}$$

$$\text{I.F.}=e^{\int\frac{2}{y}dy}=e^{2\ln y}=y^2$$

Solution is $u\cdot y^2=\int\frac{1}{y}dy+k$

$$\Rightarrow uy^2=\ln y+k \Rightarrow -\frac{e^{-2x}}{2}y^2=\ln y+k$$

$$\Rightarrow -e^{-2x}y^2=2\ln y+2k$$

$$\therefore\ e^{-2x}y^2+2\ln y=c$$

(30 - 32) : 30. (c) 31. (c) 32. (a)

$$xyp^2 - x^2p + y^2p - xy = 0$$

$$\Rightarrow xp(yp - x) + y(yp - x) = 0$$

$$\Rightarrow (yp - x)(xp + y) = 0$$

$$\Rightarrow p=\frac{x}{y} \text{ or } p=\frac{-y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{x}{y} \text{ or } \frac{dy}{dx}=\frac{-y}{x}$$

$$\Rightarrow \int ydy=\int xdx \text{ or } xdy+ydx=0$$

$$\Rightarrow \frac{y^2}{2}=\frac{x^2}{2}+c \text{ or } \int d(xy)=0$$

$$\Rightarrow y^2 - x^2 = c_1 \text{ or } xy = c_2$$

33. (A) → (p); (B) → (p); (C) → (p); (D) → (r)

(A) $xy - \log y = 1$

Differentiating w.r.t. x, we get

$$xy'+y-\frac{1}{y}.y'=0$$

$$\Rightarrow (xy - 1)y' + y^2 = 0$$

Again differentiating w.r.t. x, we get $k = 3$

(B) $P\frac{dy}{dx}=\frac{\sqrt{2}y}{\sqrt{1-x^2}}$

$\Rightarrow (1 - x^2)(y')^2 = 2y^2$

Differentiating w.r.t. x., we get

$k = 2$

(C) $F(x) = f(x)g(x)h(x)$

$\Rightarrow \log F(x) = \log f(x) + \log g(x) + \log h(x)$

$$\Rightarrow \frac{F'(x)}{F(x)}=\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}+\frac{h'(x)}{h(x)}$$

Put $x = x_0$, we get

$21 = 4 - 7 + k \Rightarrow k = 24$

(D) $(a + b)^{n-1} = a^{n-1} + b^{n-1}$

$\Rightarrow n = 2$

34. (A) → (q); (B) → (r); (C) → (s); (D) → (p)

(A) $y_2 - 8y_1 + 15y = 0$

(B) $xy_1 + y = ae^x - be^{-x} + 2x$

$xy_2 + 2y_1 = ae^x + be^{-x} + 2 = xy - x^2 + 2$

(C) $ax + byy_1 = 0 \Rightarrow \frac{yy_1}{x}=-\frac{a}{b}$

$$\Rightarrow x\left[yy_2+y_1^2\right]-yy_1=0$$

(D) $y_1 = 4a\cos(4x + b) \Rightarrow y_2 = -16a\sin(4x + b)$

$\Rightarrow y_2 + 16y = 0$

35. (A) → (r); (B) → (p); (C) → (s); (D) → (q)

(A) The given equation can be written as

$$\left(\frac{dy}{dx}-e^{-x}\right)\left(\frac{dy}{dx}-e^{x}\right)=0$$

$$\Rightarrow \frac{dy}{dx}-e^{-x}=0 \text{ or } \frac{dy}{dx}-e^{x}=0$$

$$\Rightarrow dy-e^{-x}dx=0 \text{ or } dy-e^{x}dx=0$$

$$\Rightarrow y+e^{-x}=c \text{ or } y-e^{x}=c$$

(B) $(x+y+1)dy=dx$

Put $x + y + 1 = v$

$\Rightarrow dx + dy = dv$ and the given equation reduces to $v(dv - dx) = dx$

$\Rightarrow x + c = v - \ln(v + 1)$

$\Rightarrow \ln(v + 1) = v - x - c$

or $\ln(x + y + 2) = y + 1 - c = y + c$

Also, $x + y + 2 = e^{y+c} = e^y.e^c = ce^y$

36. (4) : $2\sqrt{x}\ y' = e^{\sqrt{x}} - e^{-\sqrt{x}}$

$$\Rightarrow \ y''2\sqrt{x} + y'.2.\frac{1}{2\sqrt{x}} = \frac{y}{2\sqrt{x}}$$

$$\Rightarrow \ y''2\sqrt{x} + \frac{y'}{\sqrt{x}} = \frac{y}{2\sqrt{x}}$$

$$\Rightarrow \ 2xy'' + y' = \frac{y}{2} \Rightarrow 4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0 \ \therefore \ k = 4$$

37. (0) : By eliminating c, $y = 1$ then $\frac{dy}{dx} = 0$.

38. (2) : Put $x^2 = \sin\alpha$; $y^2 = \sin\beta$, we get $\cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$$\Rightarrow \ \cot\left(\frac{\alpha - \beta}{2}\right) = a$$

$\Rightarrow \alpha - \beta = 2\cot^{-1}(a)$

$\sin^{-1}(x^2) - \sin^{-1}(y^2) = 2\cot^{-1}(a)$

$$\Rightarrow \ \frac{1}{\sqrt{1-x^4}}(2x) - \frac{1}{\sqrt{1-y^4}}(2y)\frac{dy}{dx} = 0$$

$$\Rightarrow \ \frac{dy}{dx} = \frac{x}{y}\sqrt{\frac{1-y^4}{1-x^4}}$$

$\therefore$ order = 1, degree = 1

39. (3) : $\because \ \frac{dy}{dx} = 3x \ \Rightarrow \ dy = 3xdx$

On integrating , we get $y = \frac{3x^2}{2} + c$

Since, it passes through $(-1, -3)$ then $-3 = \frac{3}{2} + c$

$$\therefore \ c = -\frac{9}{2} \quad \therefore \ y = \frac{3x^2}{2} - \frac{9}{2} \Rightarrow 2y = 3(x^2 - 3)$$

$\therefore \ \lambda = 3$

40. (3) : $\frac{1}{y+1}dy = -\frac{\cos x}{2+\sin x}dx$

Integrating, we get $K(y + 1)(2 + \sin x) = 1$

when $x = 0$, $y = 1$, then $K = 1/4$,

$\therefore \ (y + 1)(2 + \sin x) = 4$

Now put $x = \frac{\pi}{2} \ \therefore (y+1)3 = 4 \Rightarrow y = \frac{1}{3}$

41. (3) : Given $y = C_1(\cos x \cos C_2 - \sin x \sin C_2) - C_3e^{C_4}.e^{-x} + C_5 \sin x$

$= (C_1 \cos C_2)\cos x - (C_1 \sin C_2 - C_5)\sin x - (C_3e^{C_4})e^{-x}$

$\Rightarrow \ y = l\cos x + m\sin x - ne^{-x}$,

where l, m, n are arbitrary constants

$\therefore$ order = 3

42. (9) : $a = -(2 + 1 - 1) = -2$

$b = 2\cdot1 + 1\cdot(-1) + (-1)(-2) = -1$

$c = -(2)(1)(-2) = 2$

$\therefore \ a^2 + b^2 + c^2 = 9$

43. (6) : The equation of tangent at any point $P(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$

As per given, $x - y\frac{dy}{dx} = 2x \Rightarrow -\frac{dy}{y} = \frac{dx}{x}$

By integrating , $xy = c$

By substituting $(1, 2)$, $C = 2 \ \therefore$ Equation of curve is $xy = 2$. If $y = 1/3$, then $x = 6$.

44. (5) : We have $\frac{dy}{dx} = 2x + 1 \ \Rightarrow y = x^2 + x + C$

It passes through $(1, 2) \ \therefore \ y = x^2 + x$

$$\therefore \ \text{Required area} = \int_0^1 (x^2 + x)\,dx = \frac{5}{6}\text{sq. unit}$$

So, $A = \frac{5}{6} \ \Rightarrow \ 6A = 5$

45. (2) Two unknowns, so order is 2.

(h, k) is centre

$\therefore$ Equation is $(x - h)^2 + (y - k)^2 = r^2$

Also, $2h + 3k = 5$

$$\Rightarrow \ (x-h)^2 + \left(y - \frac{5-2h}{3}\right)^2 = r^2$$

only two parameters $\Rightarrow$ order =2

46. (3) : $2x^4ydy + y^2dy + 4x^3y^2dx - x^2dx = 0$

or $\ d(x^4y^2) + y^2dy - x^2dx = 0$

47. (0) : $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1}$

$$\Rightarrow \ \frac{d^2y}{dx^2} = (-1)\left(\frac{dx}{dy}\right)^{-2}\frac{d}{dy}\left(\frac{dx}{dy}\right)\frac{dy}{dx}$$

$$= -\left(\frac{dx}{dy}\right)^{-2}.\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \ \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3\frac{d^2x}{dy^2} = 0$$

48. (2) : The given equation is reduced to

$$x = e^{xy.\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow \quad \log x = xy.\left(\frac{dy}{dx}\right) \quad \Rightarrow \quad \int y\,dy = \int \frac{1}{x}\log x\,dx$$

$$\Rightarrow \quad \frac{y^2}{2} = \frac{(\log x)^2}{2} + C$$

49. (3) : $y = e^{4x} + 2e^{-x}$; $y_1 = 4e^{4x} - 2e^{-x}$;

$y_2 = 16e^{4x} + 2e^{-x}$; $y_3 = 64e^{4x} - 2e^{-x}$

Now, $y_3 - 13y_1 = (64e^{4x} - 2e^{-x}) - 13(4e^{4x} - 2e^{-x})$

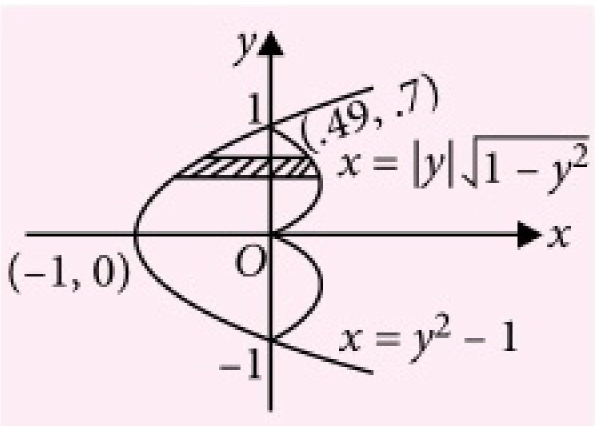
$= 12e^{4x} + 24e^{-x}$

$= 12(e^{4x} + 2e^{-x}) = 12y$

$$\therefore \quad \frac{y_3 - 13y_1}{y} = 12 \Rightarrow K = 3$$

50. (2) : $A = 2\int_0^1 \left[y\sqrt{1-y^2} - (y^2 - 1) \right] dy$

$= 2$



51. (7) : $\frac{dy}{dx} + \frac{y}{x} = x^2y^6 \Rightarrow \frac{1}{y^6}\frac{dy}{dx} + \frac{1}{xy^5} = x^2$

Put $\frac{1}{y^5} = z \Rightarrow \frac{-5}{y^6}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore \quad -\frac{1}{5}\left(\frac{dz}{dx}\right) + \frac{z}{x} = x^2 \Rightarrow \text{I.F.} = \frac{1}{x^5}$

Hence solution is $\frac{1}{x^5y^5} = \frac{5}{2x^2} + c$

52. (0) : $xy = c^2 \Rightarrow$ Diff. eq. is $x\frac{dy}{dx} + y = 0$

Replace $\frac{dy}{dx}$ with $-\frac{dx}{dy}$ and by solving it we get $x^2 - y^2 = k$

53. (1) : $y = c_1e^x.e^{c_2} + c_3e^x = (c_1.e^{c_2} + c_3)e^x = ce^x$

There is only one arbitrary constant.

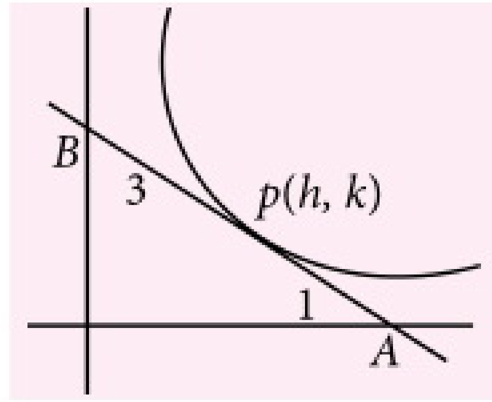
54. (2) : $(x - a)^2 + (y - a)^2 = a^2$ ...(i)

$2(x-a) + 2(y-a)y_1 = 0 \Rightarrow a = \frac{x + yy_1}{1 + y_1}$ ...(ii)

From (i) and (ii)

$$\left(x - \frac{x+yy_1}{1+y_1}\right)^2 + \left(y - \frac{x+yy_1}{1+y_1}\right)^2 = \left(\frac{x+yy_1}{1+y_1}\right)^2$$

55. (2) : $\frac{3x}{h} + \frac{y}{k} = 1$ is the equation of the tangent whose slope is $-\frac{3k}{h}$



$\frac{dy}{dx} = -\frac{3y}{x} \Rightarrow x\frac{dy}{dx} + 3y = 0$

Solving $\frac{dy}{y} = -3\frac{dx}{x} \Rightarrow \log y = -3\log x + \log c$

$\Rightarrow \quad y = cx^{-3} \Rightarrow y = \frac{c}{x^3}$, which passes through (1,1)

$\Rightarrow \quad c = 1$

$\therefore$ Equation of the curve is $y = \frac{1}{x^3}$

56. (3) : $\frac{dy}{dx} = \frac{y}{x^2} \Rightarrow \frac{dy}{y} = \frac{dx}{x^2} \Rightarrow \log y = -\frac{1}{x} + c$

$\Rightarrow \quad y = c\,e^{-\frac{1}{x}}$, which passes through (1, 3)

$\Rightarrow \quad ce^{-1} = 3 \Rightarrow c = 3e$

$\therefore \quad y = 3e^{1-\frac{1}{x}}$ is the curve

Here $f(x) = 3\,e^{1-\frac{1}{x}}$

$\Rightarrow \quad f'(x) = 3e^{1-\frac{1}{x}}\left(\frac{1}{x^2}\right) > 0 \,\forall x \in R - \{0\}$

Also $\underset{x\to\infty}{\text{Lt}} f(x) = 3e$

57. (4) : $\frac{xdy - ydx}{x^2 + y^2} = -dx \Rightarrow \frac{ydx - xdy}{x^2 + y^2} = dx$

$\Rightarrow \quad \tan^{-1}\left(\frac{x}{y}\right) = x + C$

$\Rightarrow \quad \tan^{-1}\left(\frac{x}{y}\right) = x \Rightarrow x\cot x = y$

$\Rightarrow \quad y\left(\frac{\pi}{4}\right) = \frac{\pi}{4} \quad \therefore \quad \frac{16}{\pi}y\left(\frac{\pi}{4}\right) = \frac{16}{\pi} \times \frac{\pi}{4} = 4$

MOCK TEST PAPER 2015

JEE Main & Advanced

Series-6

Differential Calculus : Chain rule, Derivatives of implicit functions and derivative of the functions defined parametrically. Second order derivative.

Matrices : Concepts of m × n (m ≤ 3, n ≤ 3), real matrices, operations of addition, scalar multiplication and multiplication of matrices. Transpose of a matrix, Determinant of a square matrix, Properties of determinants (statement only). Minor, cofactor and adjoint of a matrix, non-singular matrix, solution of system of linear equations (Not more than 3 variables).

PAPER-I

SECTION-I

Single Correct Option

This section contains 10 multiple choice questions. Each question has 4 choices, out of which ONLY ONE is correct.

1. If $f(x) = |x|$, then $f'(x)$, when $x \neq 0$ is equal to

(a) 1 (b) -1 (c) $\dfrac{|x|}{x}$ (d) 0

2. Let $f(x) = \tan 2x \cdot \tan 3x \cdot \tan 5x$, then $f'(\pi)$ equals

(a) 10 (b) -10
(c) 0 (d) none of these

3. Let $t = \dfrac{2\sqrt{2} - (1+\sqrt{3})}{\sqrt{3}-1}$ and $f(x) = \dfrac{2x}{1-x^2}$,

$g(x) = \dfrac{3x - x^3}{1-3x^2}$, then $\dfrac{d}{dt}(f(g(t)))$, $t \in R$ is

(a) 1 (b) 2
(c) 0 (d) none of these

4. If $xy = e - e^y$, then $\dfrac{d^2y}{dx^2}$ when $x = 0$ equals

(a) $1/e$ (b) $1/e^3$
(c) $1/e^2$ (d) none of these

5. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then

$\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$ equals

(a) $\cos\theta$ (b) $\sin\theta$ (c) $\sec\theta$ (d) $\text{cosec}\,\theta$

6. The derivative of function

$$f(x) = \cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x - 3\sin x)\right\}$$

$$+ \sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\sin x + 3\cos x)\right\} \text{ is}$$

(a) 0 (b) $2x$
(c) 2 (d) none of these

7. If $D_k = \begin{vmatrix} 1 & 2k & 2k-1 \\ n & n^2+n+1 & n^2 \\ n & n^2+n & n^2+n+1 \end{vmatrix}$ and

$\sum\limits_{k=1}^{n} D_k = 72$, then value of n equals

(a) 4 (b) 6 (c) 7 (d) 8

8. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & -1 & 9 \end{pmatrix}$, then the value of

$\det(\text{adj}(\text{adj}A)) =$

(a) 11 (b) 121 (c) 1331 (d) 14641

9. For the set of linear equations,
$\lambda x - 3y + z = 0$, $x + \lambda y + 3z = 1$ and
$3x + y + 5z = 2$
the value(s) of λ, for which the equations does not have unique solution, is

By : Sankar Ghosh, HOD(Math), Takshyashila. Mob : 09831244397

(a) $-1, \frac{11}{5}$ (b) $-1, -\frac{11}{5}$

(c) $-\frac{11}{5}, 1$ (d) $1, \frac{11}{5}$

10. From the matrix equation $AB = AC$, we say $B = C$ provided

(a) A is singular matrix

(b) A is square matrix

(c) A is skew symmetric matrix

(d) A is non-singular matrix

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices, out of which ONE or MORE is/are correct.

11. If $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ and $y(1) = 1$, then

(a) $y'(1) = \frac{4}{3}$ (b) $y''(1) = -\frac{4}{3}$

(c) $y''(1) = -8\frac{22}{27}$ (d) $y'(1) = \frac{2}{3}$

12. If $u = x^2 + y^2$ and $x = s + 3t$, $y = 2s - t$, then

(a) $\frac{dx}{ds} = 1$ (b) $\frac{dy}{ds} = 2$

(c) $\frac{d^2u}{ds^2} = 10$ (d) $\frac{d^2y}{ds^2} = 0$

13. If $g(x, n) = \sin^n x - \cos^n x$ and $f(x, n) = \frac{g'(-x, n)}{g(x, n-1)}$ where x measured in radians, then which of the following is not true?

(a) $g(x, 4) = g(x, 2)$

(b) $g\left(\frac{3\pi}{4}, 4\right) = 0$

(c) $f\left(\frac{\pi}{12}, 3\right) = -\cos^2\frac{\pi}{6}$

(d) $f\left(\frac{\pi}{12}, 5\right) = -1$

14. If $ad - bc \neq 0$, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^2 + xA + yI_2 = 0$, then

(a) $x = -(a + b)$ (b) $x = -(a + d)$

(c) $y = ad - bc$ (d) $y = bc - ad$

15. If

$$f(x) = \begin{vmatrix} 3 & 3x & 3x^2 + 2a^2 \\ 3x & 3x^2 + 2a^2 & 3x^3 + 6a^2x \\ 3x^2 + 2a^2 & 3x^3 + 6a^2x & 3x^4 + 12a^2x^2 + 2a^4 \end{vmatrix},$$

then

(a) $f'(x) = 0$

(b) $y = f(x)$ is a straight line parallel to x-axis

(c) $\int_0^2 f(x)dx = 32a^4$

(d) none of these

SECTION-III

Assertion & Reason Type

In each of the following questions, two statements are given. Statement-1 (Assertion) and Statement-2 (Reason). Examine the statements carefully and answer the questions according to the instruction given below.

(a) If both Statement-1 and Statement-2 are correct and Statement-2 is the proper explanation of Statement-1.

(b) If both Statement-1 and Statement-2 are correct but Statement-2 is not the proper explanation of Statement-1.

(c) If Statement-1 is correct and Statement-2 is wrong.

(d) If Statement-1 is wrong and Statement-2 is correct.

16. Let $A(\theta) = \begin{bmatrix} \cos\theta + \sin\theta & \sqrt{2}\sin\theta \\ -\sqrt{2}\sin\theta & \cos\theta - \sin\theta \end{bmatrix}$

Statement-1 : $A^3\left(\frac{\pi}{3}\right) = -I$, (where I is an identity matrix).

Statement-2 : $A(\theta)\, A(\phi) = A(\theta + \phi)$.

17. **Statement-1 :** $\begin{vmatrix} \cos\alpha & -\sin\alpha & 1 \\ \sin\alpha & \cos\alpha & 1 \\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1 \end{vmatrix}$ is independent of α.

Statement-2 : If $f(\alpha) = \lambda$, then $f(\alpha)$ is independent of α.

18. Given, $x^2 + y^2 = 1$. y' and y'' are first and second order derivatives of y respectively.

Statement-1 : $1 + yy'' + (y')^2 = 0$

Statement-2 : $xx' + 2y = 0$

19. Given, $x = f(t)$, $y = g(t)$ where t is a parameter.

Statement-1 : $\dfrac{d^2y}{dx^2} = \dfrac{g''(t)}{f''(t)}$

Statement-2 : $\dfrac{d^2y}{dx^2} = \dfrac{f'(t)g''(t) - g'(t)f''(t)}{\{f'(t)\}^3}$

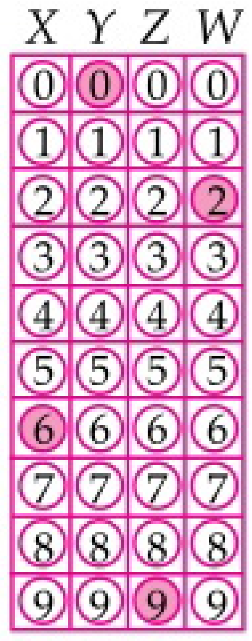
20. **Statement-1 :** If the trace of matrix $\begin{bmatrix} x-5 & 0 & 2 \\ 3 & x^2-10 & 6 \\ -2 & 3 & x-9 \end{bmatrix}$ is zero, then the values of x is -6 or 4.

Statement-2 : Distinct roots of a quadratic equation $ax^2 + bx + c = 0$ are possible if discriminant of the equation is positive *i.e.* $b^2 - 4ac > 0$.

SECTION-IV

Integer Answer Type

This section contains 5 questions. The answer to each of the questions is a single-digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to question numbers X, Y, Z and W (say) are 6, 0, 9 and 2, respectively, then the correct darkening of bubbles will look like as given.



21. If $A = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, then $A^4 = kB$. The value of k is

22. $f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$

Find the maximum value of $f(x)$.

23. Suppose $a, b, c \in R$ and $abc = 1$.

If $A = \begin{bmatrix} 2a & b & c \\ b & 2c & a \\ c & a & 2b \end{bmatrix}$ such that $A'A = 4^{1/3}I$ and $|A| > 0$, then find the value of $a^3 + b^3 + c^3$.

24. If $y = \sin(4\sin^{-1}x)$ satisfies the relation $(1 - x^2)y_2 - xy_1 + 4Ay = 0$, then A is equal to

25. If $y = \left(1 + \dfrac{1}{x}\right)^x$, then $\dfrac{2\sqrt{y_2(2) + \frac{1}{8}}}{\log\frac{3}{2} - \frac{1}{3}} =$

PAPER-II

SECTION-I

Single Option Correct

This section contains 10 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. If $f(x) = \sin\left(\dfrac{\pi}{2}[x] - x^5\right)$, $1 < x < 2$ and $[x]$ denotes the greatest integer less than or equal to x, then $f\left(\sqrt[5]{\dfrac{\pi}{2}}\right)$ is equal to

(a) $5\left(\dfrac{\pi}{2}\right)^{4/5}$ (b) $-5\left(\dfrac{\pi}{2}\right)^{4/5}$

(c) 0 (d) none of these

2. If $y = \cos^{-1}\sqrt{\dfrac{\sqrt{1+x^2}+1}{2\sqrt{1+x^2}}}$, then $\dfrac{dy}{dx}$ is equal to

(a) $\dfrac{1}{1+x^2}$ (b) $\dfrac{1}{1-x^2}$

(c) $\dfrac{1}{2(1+x^2)}$ (d) none of these

3. If $f(x)=\left(\frac{a+x}{b+x}\right)^{a+b+2x}$, then $f'(0)$ is equal to

(a) $\left(2\log\left(\frac{a}{b}\right)+\frac{a^2-b^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$

(b) $\left(2\log\left(\frac{a}{b}\right)+\frac{b^2-a^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$

(c) $\left(2\log\left(\frac{a}{b}\right)+\frac{a^2+b^2}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$

(d) none of these

4. If $y=\begin{vmatrix}\sin x & \cos x & \sin x\\ \cos x & -\sin x & \cos x\\ x & 1 & 1\end{vmatrix}$, then $\frac{dy}{dx}=$

(a) 1 (b) –1
(c) 0 (d) none of these

5. Given : $x=a\cos t\sqrt{\cos 2t}$, $y=a\sin t\sqrt{\cos 2t}$ $(a>0)$, then $\left|\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}\Big/\frac{d^2y}{dx^2}\right|$ at $\frac{\pi}{6}$ is given by

(a) $\frac{a}{3}$ (b) $a\sqrt{2}$
(c) $\frac{\sqrt{2}}{3a}$ (d) $\frac{\sqrt{2}a}{3}$

6. If $f(x)=|x|^{|\tan x|}$, then $f'\left(-\frac{\pi}{6}\right)$ is equal to

(a) $\left(\frac{\pi}{6}\right)^{1/3}\left\{\frac{2\sqrt{3}}{\pi}-\frac{4}{3}\log\frac{6}{\pi}\right\}$

(b) $\left(\frac{\pi}{6}\right)^{1/\sqrt{3}}\left\{\frac{-2\sqrt{3}}{\pi}+\frac{4}{3}\log\frac{6}{\pi}\right\}$

(c) $\left(\frac{\pi}{6}\right)^{1/\sqrt{3}}\left\{\frac{2\sqrt{3}}{\pi}+\frac{4}{3}\log\frac{6}{\pi}\right\}$

(d) none of these

7. If $[x]$ stands for the greatest integer less or equal to x, then in order that set of equations $x-3y=4$, $5x+y=2$, $[2\pi]x-[e]y=[2a]$ may be consistent, then a should lie in

(a) $\left[3,\frac{7}{2}\right)$ (b) $\left(3,\frac{7}{3}\right)$
(c) $\left(3,\frac{7}{3}\right]$ (d) none of these

8. The values λ and μ for which the equations $x+y+z=3$, $x+3y+2z=6$ and $x+\lambda y+3z=\mu$ have

(a) a unique solution; if $\lambda=5, \mu\in R$
(b) no solution; if $\lambda\neq 5, \mu=9$
(c) infinitely many solution; if $\lambda=5, \mu\neq 9$
(d) none of these

9. A skew symmetric matrix S satisfied the relation $S^2+I=0$, where I is a unit matrix. Then SS' is equal to

(a) I (b) $2I$
(c) $-I$ (d) none of these

10. If A and B are square matrices of the same order and A is non-singular, then for a positive integer n, $(A^{-1}BA)^n$ is equal to

(a) $A^{-n}B^nA^n$ (b) $A^nB^nA^{-n}$
(c) $A^{-1}B^nA$ (d) $n(A^{-1}BA)$

SECTION-II

Multiple Correct Option

This section contains 5 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which ONE OR MORE is/are correct.

11. If $f(x)=\sin^{-1}(\sin x)$, then $f'(x)$ is

(a) $1\ \forall\ x$ (b) $-1\ \forall\ x$
(c) –1 in 2nd and 3rd quadrant
(d) 1 in 4th quadrant

12. If $f(x)=x^{\frac{1}{\log x}}$, then $f'(x)=0$ for $x=$

(a) 0 (b) 1
(c) 2 (d) e

13. If $x^y=y^x$, then $y'(e)=$

(a) 0 (b) –1
(c) 1 (d) non-existent

14. The determinant $\begin{vmatrix}a & b & a\alpha+b\\ b & c & b\alpha+c\\ a\alpha+b & b\alpha+c & 0\end{vmatrix}=0$ if

(a) a, b, c are in A.P.

(b) a, b, c are in G.P.
(c) α is a root of $ax^2 + bx + c = 0$
(d) $(x - \alpha)$ is a factor of $ax^2 + 2bx + c = 0$

15. If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, then $A^{-1} =$

(a) A^T (b) A
(c) $A^2 + A - I$ (d) $A^2 - A - I$

SECTION-III

Linked Comprehension Type

This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE is/are correct.

Paragraph for Question 16 to 18

If $u = e^x \sin x$ and $v = e^x \cos x$, then

16. The value of $v\dfrac{du}{dx} - u\dfrac{dv}{dx}$ is

(a) $u^2 + v^2$ (b) $u + v$
(c) $u^2 - v^2$ (d) $u - v$

17. The value of $\dfrac{d^2u}{dx^2}$ is

(a) $2u$ (b) $2v$ (c) u/v (d) v/u

18. The value of $\dfrac{d^2v}{dx^2}$ is

(a) $2u$ (b) $2v$ (c) $-2u$ (d) $-2v$

Paragraph for Question 19 to 21

Let A be $n \times n$ matrix with determinant $|A| \neq 0$, then

19. $|\text{adj } A| =$

(a) $|A|$ (b) $|A|^n$
(c) $|A|^{n-1}$ (d) $|A|^{n+1}$

20. $|\text{adj}A|^{-1} =$

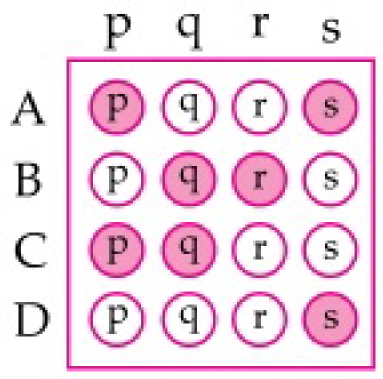
(a) A (b) $\dfrac{A}{|A|}$
(c) $\dfrac{A}{|A|^n}$ (d) $\dfrac{A}{|A|^{n-1}}$

21. $|\text{adj (adj } A)| =$

(a) $|A|^{2n}$ (b) $|A|^{n^2}$
(c) $|A|^{(n-1)^2}$ (d) $|A|^{(n+1)^2}$

SECTION-IV

Matrix Match Type



This section contains 2 questions. Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s, then the correct darkening of bubbles will look like the above.

22. Match the following:

	Column I		Column II
(A)	If $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$, then X will be	(p)	$\begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix}$
(B)	If $A + I = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, then $(A + I)(A - I) =$	(q)	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(C)	$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$ is equal to	(r)	$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$
(D)	If $f(x) = x^2 - 5x$ and $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $f(A)$ is equal to	(s)	$\begin{bmatrix} -5 & -4 \\ 8 & -9 \end{bmatrix}$

23. Match the following:

	Column I		Column II
(A)	$xy - \log y = 1$	(p)	$xy_2 + \frac{1}{2}y_1 - \frac{1}{4}y = 0$
(B)	$y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$	(q)	$(1 - x^2)y_1 - xy = 1$

(C)	$y=\sqrt{2x-x^2}$	(r)	$y^2+(xy-1)y_1=0$
(D)	$y=e^{\sqrt{x}}+e^{-\sqrt{x}}$	(s)	$y^3y_2+1=0$

SOLUTIONS

PAPER-I

1. (c): $f(x)=\begin{cases}-x, \text{ when } x<0\\ x, \text{when } x\geq 0\end{cases}$

$\therefore\ f'(x)=\begin{cases}-1 \text{ if } x<0, & i.e. \dfrac{|x|}{x} \text{ if } x<0\\ 1 \text{ if } x>0, & i.e. \dfrac{|x|}{x} \text{ if } x>0\end{cases}$

2. (c) : $f(x) = \tan 2x \tan 3x \tan 5x$

$\therefore\ f'(x)=2\sec^2 2x\tan 3x\tan 5x+\tan 2x(3\sec^2 3x)\times \tan 5x + \tan 2x \tan 3x(5\sec^2 5x)$

$\therefore\ f'(\pi) = 0\quad [\because\ \tan n\pi = 0\ \forall\ n \in Z]$

3. (c): $t=\dfrac{2\sqrt{2}-(1+\sqrt{3})}{\sqrt{3}-1}=\dfrac{\{2\sqrt{2}-(1+\sqrt{3})\}(\sqrt{3}+1)}{2}$

$=\dfrac{(2\sqrt{6}+2\sqrt{2})-(3+1+2\sqrt{3})}{2}$

$=\dfrac{2\sqrt{6}+2\sqrt{2}-4-2\sqrt{3}}{2}$

$\Rightarrow\ t=\sqrt{6}+\sqrt{2}-\sqrt{3}-2=\tan\left(7\frac{1}{2}\right)^\circ$

Now, $g(x)=\dfrac{3x-x^3}{1-3x^2}$

$\Rightarrow\quad g(t)=\dfrac{3\tan\left(7\frac{1}{2}\right)^\circ-\tan^3\left(7\frac{1}{2}\right)^\circ}{1-3\tan^2\left(7\frac{1}{2}\right)^\circ}$

$\Rightarrow\ g(t)=\tan 3\left(7\frac{1}{2}\right)^\circ=\tan\left(22\frac{1}{2}\right)^\circ=\sqrt{2}-1$

Now, $f(g(t))=f(\sqrt{2}-1)=\dfrac{2(\sqrt{2}-1)}{1-(\sqrt{2}-1)^2}$

$=\dfrac{2(\sqrt{2}-1)}{1-3+2\sqrt{2}}=\dfrac{2(\sqrt{2}-1)}{2(\sqrt{2}-1)}$

$\therefore\ f(g(t))=1\ \Rightarrow\ \dfrac{d}{dt}(f(g(t)))\ =0$

4. (c) : $xy = e - e^y\ \Rightarrow\ x\dfrac{dy}{dx}+y=-e^y\dfrac{dy}{dx}$

$\Rightarrow\ (x+e^y)\dfrac{dy}{dx}=-y$ ... (i)

When $x = 0$, then $y = 1$

$\therefore$ (i) becomes $\dfrac{dy}{dx}=-\dfrac{1}{e}$

Again differentiating (i) w.r.t. x, we get

$(x+e^y)\dfrac{d^2y}{dx^2}+\left(1+e^y\dfrac{dy}{dx}\right)\dfrac{dy}{dx}=-\dfrac{dy}{dx}$

$\Rightarrow\ (x+e^y)\dfrac{d^2y}{dx^2}+\left(2+e^y\dfrac{dy}{dx}\right)\dfrac{dy}{dx}=0$

Now using $x = 0$, $y = 1$ and $\dfrac{dy}{dx}=-\dfrac{1}{e}$, we get

$e\dfrac{d^2y}{dx^2}-\left(2-\dfrac{e}{e}\right)\dfrac{1}{e}=0$

$\Rightarrow\ e\dfrac{d^2y}{dx^2}-\dfrac{1}{e}=0\ \Rightarrow\ \dfrac{d^2y}{dx^2}=\dfrac{1}{e^2}$

5. (c) : We have $x = a\cos^3\theta$, $y = a\sin^3\theta$

$\therefore\ \dfrac{dx}{d\theta}=-3a\cos^2\theta\sin\theta,\ \dfrac{dy}{d\theta}=3a\sin^2\theta\cos\theta$

$\therefore\ \dfrac{dy}{dx}=-\tan\theta\ \Rightarrow\ \left(\dfrac{dy}{dx}\right)^2=\tan^2\theta$

$\therefore\ 1+\left(\dfrac{dy}{dx}\right)^2=\sec^2\theta \Rightarrow \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}=\sec\theta$

6. (c) : Let $\dfrac{2}{\sqrt{13}}=\cos\theta$ and $\dfrac{3}{\sqrt{13}}=\sin\theta$

$\therefore\ f(x) = \cos^{-1}(\cos\theta\cos x - \sin\theta\sin x) + \sin^{-1}(\cos\theta\sin x + \sin\theta\cos x)$

$= \cos^{-1}(\cos(\theta + x)) + \sin^{-1}(\sin(\theta + x))$

$\Rightarrow\ f(x) = 2x + 2\theta\ \therefore f'(x) = 2$

7. (d): $D_k=\begin{vmatrix}1 & 2k & 2k-1\\ n & n^2+n+1 & n^2\\ n & n^2+n & n^2+n+1\end{vmatrix}$

$\therefore\ \sum_{k=1}^{n}D_k=\begin{vmatrix}\sum_{k=1}^{n}1 & \sum_{k=1}^{n}2k & \sum_{k=1}^{n}2k-1\\ n & n^2+n+1 & n^2\\ n & n^2+n & n^2+n+1\end{vmatrix}$

$$= \begin{vmatrix} n & n(n+1) & n^2 \\ n & n^2+n+1 & n^2 \\ n & n(n+1) & n^2+n+1 \end{vmatrix}$$

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= \begin{vmatrix} 0 & -1 & 0 \\ 0 & 1 & -(n+1) \\ n & n(n+1) & n^2+n+1 \end{vmatrix}$$

$$\therefore \quad \sum_{k=1}^{n} D_k = n(n+1)$$

$\therefore \quad n(n+1) = 72 = 8(8+1) \Rightarrow n = 8$

8. (d) : $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 3 & -1 & 9 \end{pmatrix}$

$|A| = 1(9 + 2) = 11 \neq 0$

We know, if n is the order of a matrix B and $|B| \neq 0$, then $\text{adj}(\text{adj } B) = |B|^{n-2}B$

$\therefore \quad \text{adj}(\text{adj } A) = |A|^{3-2}A \quad (\because n = 3)$

$\Rightarrow \quad |\text{adj}(\text{adj } A)| = ||A|\, A|$

$= |A|^3 \det A \quad [\because \det(kA) = k^n \det A]$

$= |A|^4 = (11)^4 = 14641$

9. (a) : As system have no unique solution.

$\therefore$ System either have no solution or infinite solution.

$\therefore \quad |A| = 0$

$$\Rightarrow \begin{vmatrix} \lambda & -3 & 1 \\ 1 & \lambda & 3 \\ 3 & 1 & 5 \end{vmatrix} = 0 \Rightarrow 5\lambda^2 - 6\lambda - 11 = 0$$

$\Rightarrow \quad (5\lambda - 11)(\lambda + 1) = 0$

$\therefore \quad \lambda = -1, \dfrac{11}{5}$

10. (d) : Let $|A| \neq 0 \quad \therefore A^{-1}$ exists.

Given that $AB = AC$

$\therefore \quad A^{-1}(AB) = A^{-1}(AC)$

$\Rightarrow \quad B = C$ (by associative property)

$\Rightarrow \quad A$ is non-singular.

11. (a, c) : Given, $x^3 - 2x^2y^2 + 5x + y - 5 = 0$

$$\Rightarrow \quad 3x^2 - 2\left(2x^2y\frac{dy}{dx} + 2xy^2\right) + 5 + \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3x^2 - 4x^2y\frac{dy}{dx} - 4xy^2 + 5 + \frac{dy}{dx} = 0$$

$$\Rightarrow \quad (1 - 4x^2y)\frac{dy}{dx} = 4xy^2 - 3x^2 - 5 \qquad \text{... (i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{4xy^2 - 3x^2 - 5}{1 - 4x^2y}$$

When $x = 1, y = 1$, then $\dfrac{dy}{dx} = \dfrac{4-3-5}{1-4} = \dfrac{-4}{-3} = \dfrac{4}{3}$

$\therefore \quad y'(1) = \dfrac{4}{3}$

Again, diff. (i) with respect to x, we get

$$(1 - 4x^2y)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(-4x^2\frac{dy}{dx} - 8xy\right) = 4\left(2xy\frac{dy}{dx} + y^2\right) - 6x$$

Now putting $x = 1$, $y = 1$ and $\dfrac{dy}{dx} = \dfrac{4}{3}$ in the above equation, we get

$$(1-4)\frac{d^2y}{dx^2} - 4\cdot 1^2\left(\frac{4}{3}\right)^2 - 8\cdot 1\cdot 1\cdot\frac{4}{3} = 4\left(2\cdot 1\cdot 1\cdot\frac{4}{3} + 1^2\right) - 6\cdot 1$$

$$\Rightarrow \quad -3\frac{d^2y}{dx^2} - \frac{64}{9} - \frac{32}{3} = \frac{32}{3} + 4 - 6$$

$$\Rightarrow \quad -3\frac{d^2y}{dx^2} = \frac{64}{9} + \frac{32}{3} + \frac{32}{3} - 2 = \frac{238}{9}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{238}{27} = -8\frac{22}{27}$$

12. (a, b, c, d) : Given $u = x^2 + y^2$ and $x = s + 3t$, $y = 2s - t$

$$\therefore \quad \frac{dx}{ds} = 1, \ \frac{dy}{ds} = 2 \qquad \therefore \ \frac{d^2y}{ds^2} = 0$$

Again, $u = (s + 3t)^2 + (2s - t)^2$

$= s^2 + 9t^2 + 6st + 4s^2 + t^2 - 4st$

$$\Rightarrow \quad u = 5s^2 + 10t^2 + 2st \Rightarrow \frac{du}{ds} = 10s + 2t$$

$$\therefore \quad \frac{d^2u}{ds^2} = 10$$

13. (c, d) : Here $g(x, n) = \sin^n x - \cos^n x$

and $f(x, n) = \dfrac{g'(-x, n)}{g(x, n-1)}$

$g(x, 4) = \sin^4 x - \cos^4 x = (\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)$

$= \sin^2 x - \cos^2 x$

$\Rightarrow\ g(x, 4) = g(x, 2)\quad \therefore$ Option (a) is correct.

$$g\left(\frac{3\pi}{4}, 4\right) = \sin^4\frac{3\pi}{4} - \cos^4\frac{3\pi}{4}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^4 - \left(-\frac{1}{\sqrt{2}}\right)^4$$

$\therefore\ g\left(\dfrac{3\pi}{4}, 4\right) = 0\ \therefore$ Option (b) is correct.

Now, $f(x, n) = \dfrac{g'(-x, n)}{g(x, n-1)} = \dfrac{g'(x, n)_{x=-x}}{g(x, n-1)}$

$$= \frac{(n\sin^{n-1}x\cos x + n\cos^{n-1}x\sin x)_{x=-x}}{\sin^{n-1}x - \cos^{n-1}x}$$

$$\therefore\ f(x, n) = \frac{n(-1)^{n-1}\sin^{n-1}x\cos x - n\cos^{n-1}x\sin x}{\sin^{n-1}x - \cos^{n-1}x}$$

$$\therefore\ f(x, 3) = \frac{3(\sin^2 x\cos x - \cos^2 x\sin x)}{\sin^2 x - \cos^2 x}$$

$$= \frac{3\sin x\cos x(\sin x - \cos x)}{(\sin x + \cos x)(\sin x - \cos x)}$$

$$= \frac{3}{2}\left(\frac{\sin 2x}{\sin x + \cos x}\right)$$

$$\therefore\ f\left(\frac{\pi}{12}, 3\right) = \frac{3}{2}\sin\frac{\pi}{6} \times \frac{1}{\sin 15° + \cos 15°} \neq -\cos^2\frac{\pi}{6}$$

Also, $f\left(\dfrac{\pi}{12}, 5\right) \neq -1$

14. (b, c) : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc \neq 0$

Now, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$

Given, $A^2 + xA + yI_2 = 0$

$$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} + x\begin{bmatrix} a & b \\ c & d \end{bmatrix} + y\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} a^2+bc+xa+y & ab+bd+xb \\ ac+cd+xc & bc+d^2+xd+y \end{bmatrix} = 0$$

$\Rightarrow\ a^2 + bc + xa + y = 0$ and $ab + bd + xb = 0$

$\therefore\ x = -(a + d)$ and $y = ad - bc$

15. (a, b) :

$$f(x) = \begin{vmatrix} 3 & 3x & 3x^2+2a^2 \\ 3x & 3x^2+2a^2 & 3x^3+6a^2x \\ 3x^2+2a^2 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$$

$$= \frac{1}{x^3}\begin{vmatrix} 3x^2 & 3x^3 & 3x^4+2a^2x^2 \\ 3x^2 & 3x^3+2a^2x & 3x^4+6a^2x^2 \\ 3x^2+2a^2 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$$

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= \frac{1}{x^3}\begin{vmatrix} 0 & -2a^2x & -4a^2x^2 \\ -2a^2 & -4a^2x & -6a^2x^2-2a^4 \\ 3x^2+2a^2 & 3x^3+6a^2x & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$$

$$= \frac{1}{2x^4}\begin{vmatrix} 0 & -4a^2x^2 & -4a^2x^2 \\ -2a^2 & -8a^2x^2 & -6a^2x^2-2a^4 \\ 3x^2+2a^2 & 6x^4+12a^2x^2 & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$$

On applying $C_2 \to C_2 - C_3$, we get

$$= \frac{1}{2x^4}\begin{vmatrix} 0 & 0 & -4a^2x^2 \\ -2a^2 & -2a^2x^2+2a^4 & -6a^2x^2-2a^4 \\ 3x^2+2a^2 & 3x^4-2a^4 & 3x^4+12a^2x^2+2a^4 \end{vmatrix}$$

$$= \frac{1}{2x^4}(-4a^2x^2)[-6a^2x^4+4a^6+6a^2x^4-2x^2a^4-4a^6]$$

$$= \frac{-4a^2x^2 \times -2x^2a^4}{2x^4} = 4a^6$$

$\Rightarrow\ f(x) = 4a^6\quad \therefore\ f'(x) = 0$

Since $f'(x) = 0\ \therefore\ f(x)$ is a straight line parallel to x-axis.

16. (b) : $A^2(\theta)=\begin{bmatrix}\cos\theta+\sin\theta & \sqrt{2}\sin\theta\\ -\sqrt{2}\sin\theta & \cos\theta-\sin\theta\end{bmatrix}\times$

$$\begin{bmatrix}\cos\theta+\sin\theta & \sqrt{2}\sin\theta\\ -\sqrt{2}\sin\theta & \cos\theta-\sin\theta\end{bmatrix}$$

$$=\begin{bmatrix}(\cos\theta+\sin\theta)^2-2\sin^2\theta & \sqrt{2}\sin\theta(\cos\theta+\sin\theta)+\sqrt{2}\sin\theta(\cos\theta-\sin\theta)\\ -\sqrt{2}\sin\theta(\cos\theta+\sin\theta)-\sqrt{2}\sin\theta(\cos\theta-\sin\theta) & -2\sin^2\theta+(\cos\theta-\sin\theta)^2\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\theta+\sin 2\theta & \sqrt{2}\sin 2\theta\\ -\sqrt{2}\sin 2\theta & \cos 2\theta-\sin 2\theta\end{bmatrix}$$

$$\therefore\quad A^3(\theta)=\begin{bmatrix}\cos 3\theta+\sin 3\theta & \sqrt{2}\sin 3\theta\\ -\sqrt{2}\sin 3\theta & \cos 3\theta-\sin 3\theta\end{bmatrix}$$

Now, $A^3\left(\frac{\pi}{3}\right)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}=-I$

Again $A(\theta)A(\phi)=\begin{bmatrix}\cos\theta+\sin\theta & \sqrt{2}\sin\theta\\ -\sqrt{2}\sin\theta & \cos\theta-\sin\theta\end{bmatrix}\times$

$$\begin{bmatrix}\cos\phi+\sin\phi & \sqrt{2}\sin\phi\\ -\sqrt{2}\sin\phi & \cos\phi-\sin\phi\end{bmatrix}$$

$$=\begin{bmatrix}\cos(\theta+\phi)+\sin(\theta+\phi) & \sqrt{2}\sin(\theta+\phi)\\ -\sqrt{2}\sin(\theta+\phi) & \cos(\theta+\phi)-\sin(\theta+\phi)\end{bmatrix}$$

$= A(\theta+\phi)$

Thus both statement-1 and statement-2 are correct but statement-2 is not the correct explanation of statement-1.

17. (a) : Let $f(\alpha)=\begin{vmatrix}\cos\alpha & -\sin\alpha & 1\\ \sin\alpha & \cos\alpha & 1\\ \cos(\alpha+\beta) & -\sin(\alpha+\beta) & 1\end{vmatrix}$

On applying $R_3\to R_3-R_1(\cos\beta)+R_2(\sin\beta)$, we get

$$f(\alpha)=\begin{vmatrix}\cos\alpha & -\sin\alpha & 1\\ \sin\alpha & \cos\alpha & 1\\ 0 & 0 & 1+\sin\beta-\cos\beta\end{vmatrix}$$

$= (1+\sin\beta-\cos\beta)(\cos^2\alpha+\sin^2\alpha)$

$\therefore\quad f(\alpha)=1+\sin\beta-\cos\beta$, which is independent of α.

18. (c) : Given, $x^2+y^2=1$ ... (i)

Diff. (i) w.r.t. x, we get,

$2x+2yy'=0 \Rightarrow x+yy'=0$

$\Rightarrow 1+yy''+(y')^2=0$

Thus, statement-1 is true.

Again, diff. (i) w.r.t. y, we get

$$\frac{d}{dy}(x^2)+\frac{d}{dy}(y^2)=0 \Rightarrow 2xx'+2y=0$$

$\Rightarrow xx'+y=0$

Thus statement-2 is false.

19. (d) : Given, $x=f(t)$ and $y=g(t)$

$$\therefore\quad \frac{dx}{dt}=f'(t)\ \text{ and }\ \frac{dy}{dt}=g'(t)$$

$$\Rightarrow \frac{dy}{dx}=\frac{g'(t)}{f'(t)}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{g'(t)}{f'(t)}\right)=\frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)\cdot\frac{dt}{dx}$$

$$=\frac{f'(t)g''(t)-g'(t)f''(t)}{\{f'(t)\}^2}\times\frac{1}{f'(t)}$$

$$\text{or}\quad \frac{d^2y}{dx^2}=\frac{f'(t)g''(t)-g'(t)f''(t)}{\{f'(t)\}^3}$$

20. (b) : The given matrix is

$$\begin{bmatrix}x-5 & 0 & 2\\ 3 & x^2-10 & 6\\ -2 & 3 & x-9\end{bmatrix}$$

Since the trace of the given matrix is zero

$\therefore\quad x-5+x^2-10+x-9=0$

$\Rightarrow x^2+2x-24=0\ \therefore\ (x-4)(x+6)=0$

$\Rightarrow x=4,-6$

Thus, both the statements are true but the statement-2 is not the correct explanation of statement-1.

21. (8) : Given, $A=\begin{bmatrix}i & -i\\ -i & i\end{bmatrix}$ and $B=\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$

$$A^2=\begin{bmatrix}i & -i\\ -i & i\end{bmatrix}\begin{bmatrix}i & -i\\ -i & i\end{bmatrix}=\begin{bmatrix}i^2+i^2 & -i^2-i^2\\ -i^2-i^2 & i^2+i^2\end{bmatrix}$$

$$A^2=\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix}\Rightarrow A^4=\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix}\begin{bmatrix}-2 & 2\\ 2 & -2\end{bmatrix}$$

$$\therefore \quad A^4 = 8\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 8B$$

$\therefore \quad k = 8.$

22. (6) : $$f(x) = \begin{vmatrix} 1+\sin^2 x & \cos^2 x & 4\sin 2x \\ \sin^2 x & 1+\cos^2 x & 4\sin 2x \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

On applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \sin^2 x & \cos^2 x & 1+4\sin 2x \end{vmatrix}$$

On applying $C_2 \to C_2 + C_1$, we get

$$f(x) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ \sin^2 x & 1 & 1+4\sin 2x \end{vmatrix} = 1(1+4\sin 2x+1)$$

$\therefore \quad f(x) = 2 + 4\sin 2x$

$f(x)$ will be maximum when $\sin 2x = 1$

$\therefore$ The maximum value of $f(x)$ is 6.

23. (4) : $$A = \begin{bmatrix} 2a & b & c \\ b & 2c & a \\ c & a & 2b \end{bmatrix} \quad \therefore A' = \begin{bmatrix} 2a & b & c \\ b & 2c & a \\ c & a & 2b \end{bmatrix}$$

$\therefore \quad |A'A| = \det(4^{1/3}I) = 4\det(I) = 4$

$\Rightarrow \quad [\det(A)]^2 = 4 \;\Rightarrow\; \det(A) = \pm 2$

Now, $|A| = 2a(4bc - a^2) - b(2b^2 - ac) + c(ab - 2c^2)$

$= 10abc - 2(a^3 + b^3 + c^3)$

$\therefore \quad 2 = 10 - 2(a^3 + b^3 + c^3) \qquad [\because |A| > 0]$

$\Rightarrow \quad 2(a^3 + b^3 + c^3) = 8 \;\Rightarrow\; a^3 + b^3 + c^3 = 4$

24. (4) : $y = \sin(4\sin^{-1}x)$

$$\Rightarrow \quad y_1 = \cos(4\sin^{-1}x) \times \frac{4}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}\, y_1 = 4\cos(4\sin^{-1}x)$$

$\Rightarrow \quad (1 - x^2)y_1^2 = 16\{1 - \sin^2(4\sin^{-1}x)\}$

$\Rightarrow \quad (1 - x^2)y_1^2 = 16 - 16y^2$

$\Rightarrow \quad (1 - x^2)2y_1y_2 + y_1^2(-2x) = -32yy_1$

$\Rightarrow \quad (1 - x^2)y_2 - xy_1 = -16y$

$\Rightarrow \quad (1 - x^2)y_2 - xy_1 + 16y = 0$

$\therefore \quad A = 4$

25. (3) : $$y = \left(1 + \frac{1}{x}\right)^x \qquad \text{...(i)}$$

$$\therefore \quad y(2) = \left(\frac{3}{2}\right)^2$$

Now differentiating (i) with respect to x, we get

$$y_1 = \left[\log\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right]y \qquad \text{... (ii)}$$

$$\therefore \quad y_1(2) = \left(\log\frac{3}{2} - \frac{1}{3}\right)\left(\frac{3}{2}\right)^2 \quad \left(\because y(2) = \left(\frac{3}{2}\right)^2\right)$$

Differentiating (ii) with respect to x, we get

$$y_2 = \left[\log\left(1+\frac{1}{x}\right) - \frac{1}{1+x}\right]y_1 + y\left[\frac{1}{1+\frac{1}{x}}\left(\frac{-1}{x^2}\right) + \frac{1}{(1+x)^2}\right]$$

$$\therefore \quad y_2(2) = \left(\log\frac{3}{2} - \frac{1}{3}\right)\left(\log\frac{3}{2} - \frac{1}{3}\right)\left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2\left(\frac{1}{9} - \frac{1}{6}\right)$$

$$= \left(\log\frac{3}{2} - \frac{1}{3}\right)^2\left(\frac{3}{2}\right)^2 - \frac{1}{18} \times \frac{9}{4}$$

$$= \left\{\left(\log\frac{3}{2} - \frac{1}{3}\right)\left(\frac{3}{2}\right)\right\}^2 - \frac{1}{8}$$

$$\therefore \quad \sqrt{y_2(2) + \frac{1}{8}} = \left(\log\frac{3}{2} - \frac{1}{3}\right)\left(\frac{3}{2}\right)$$

Now, $$\frac{2\sqrt{y_2(2) + \frac{1}{8}}}{\log\frac{3}{2} - \frac{1}{3}} = \frac{2\left(\log\frac{3}{2} - \frac{1}{3}\right)\left(\frac{3}{2}\right)}{\left(\log\frac{3}{2} - \frac{1}{3}\right)} = 2 \times \frac{3}{2} = 3$$

PAPER-II

1. (c)	**2.** (c)	**3.** (b)	**4.** (a)
5. (d)	**6.** (b)	**7.** (a)	**8.** (d)
9. (a)	**10.** (c)	**11.** (c, d)	**12.** (c, d)
13. (b, c)	**14.** (b, d)	**15.** (b, c)	**16.** (a)
17. (b)	**18.** (c)	**19.** (c)	**20.** (b)
21. (c)	**22.** A $\to$ r, B $\to$ s, C $\to$ q, D $\to$ p		
23. A $\to$ r, B $\to$ q, C $\to$ s, D $\to$ p			

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If A is a square matrix such that

$$A(\text{Adj}A)=\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}, \text{ then } \frac{|\text{Adj}(\text{Adj } A)|}{|\text{Adj } A|} \text{ is}$$

equal to

(a) 256 (b) 64
(c) 32 (d) 16

2. If two distinct chords of a parabola $y^2 = 4ax$, passing through $(a, 2a)$ are bisected by the line $x + y = 1$, then length of latus rectum can be

(a) 2 (b) 4
(c) 5 (d) 6

3. The vectors

$$(al+a'l')\hat{i}+(am+a'm')\hat{j}+(an+a'n')\hat{k},$$

$$(bl+b'l')\hat{i}+(bm+b'm')\hat{j}+(bn+b'n')\hat{k},$$

$$(cl+c'l')\hat{i}+(cm+c'm')\hat{j}+(cn+c'n')\hat{k}$$

(a) form an equilateral triangle
(b) are coplanar
(c) are collinear
(d) are mutually perpendicular

4. $f(x)=\cos x-\int_0^x (x-t)f(t)\,dt$, then $f''(x)+f(x)$ is equal to

(a) $-\cos x$ (b) $-\sin x$
(c) $\int_0^x (x-t)f(t)dt$ (d) zero

5. If $f(x)=\int_1^x \frac{\tan^{-1} t}{t}dt,\ x>0$, then the value of $f(e^2)-f\left(\frac{1}{e^2}\right)$ is

(a) 0 (b) $\frac{\pi}{2}$
(c) π (d) 2π

6. The number of solutions of the equation $\tan^{-1}\left(\frac{x}{1-x^2}\right)+\tan^{-1}\left(\frac{1}{x^3}\right)=\frac{3\pi}{4}$, belonging to the interval $(0, 1)$ is

(a) 0 (b) 1
(c) 2 (d) infinite

7. With respect to a variable point on the line $x + y = 2a$, chord of contact of the circle $x^2 + y^2 = a^2$ is drawn. If it passes through a fixed point F, the chord of the circle with F as mid point is

(a) parallel to the line $x + y = 2a$
(b) perpendicular to the line $x + y = 2a$
(c) makes angle 45° with the line $x + y = 2a$
(d) none of these

8. The area of the region bounded by the curves $|y+x|\le 1, |y-x|\le 1$ and $3x^2+3y^2=1$ is

(a) $\left(1-\frac{\pi}{3}\right)$ sq. units (b) $\left(2-\frac{\pi}{3}\right)$ sq. units
(c) $\left(3-\frac{\pi}{3}\right)$ sq. units (d) none of these

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9. Solution of the differential equation

$$\frac{dy}{dx}+\frac{y}{x}=\frac{1}{(1+\log x+\log y)^2} \quad \text{is}$$

(a) $xy(1+(\ln xy)^2)=\frac{x^2}{2}+C$

(b) $xy(1+\ln xy)=\frac{x^2}{2}+C$

(c) $xy(1+\ln xy)=\frac{x}{2}+C$

(d) none of these

10. $P(t^2, 2t)$, $t \in (0, 1]$ is any arbitrary point on $y^2 = 4x$. 'Q' is the foot of perpendicular drawn from focus 'S' to the tangent drawn at P. Maximum area of triangle PQS is

(a) 1 sq. unit (b) 2 sq. units

(c) $\frac{1}{2}$ sq. unit (d) 4 sq. units

SOLUTIONS

1. (d) $A(\text{Adj}A) = |A|.I_n$

Clearly $|A| = 4$

$n = 3$

$|\text{Adj}(\text{Adj}A)| = |A|^{(n-1)^2} = 4^4 = 256$

$|\text{Adj}A| = |A|^{n-1} = 4^2 = 16.$

$$\therefore \frac{|\text{Adj}(\text{Adj }A)|}{|\text{Adj }A|}=\frac{256}{16}=16.$$

2. (a) : Any point on the line $x + y = 1$ can be taken as $(t, 1 - t)$

Equation of chord with this as mid point is

$y(1 - t) - 2a(x + t) = (1 - t)^2 - 4at$

It passes through $(a, 2a)$

$\Rightarrow t^2 - 2t + 2a^2 - 2a + 1 = 0$

This should have two distinct real roots so

$a^2 - a < 0$

$\Rightarrow 0 < a < 1$

$\Rightarrow$ length of latus rectum < 4.

3. (b) : $$\Delta=\begin{vmatrix} al+a'l' & am+a'm' & an+a'n' \\ bl+b'l' & bm+b'm' & bn+b'n' \\ cl+c'l' & cm+c'm' & cn+c'n' \end{vmatrix}$$

$$=\begin{vmatrix} a & a' & 0 \\ b & b' & 0 \\ c & c' & 0 \end{vmatrix}\times\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix}=0.$$

4. (a) : $$f'(x)=-\sin x-\left[xf(x)+\int_0^x f(t)dt\right]+xf(x)$$

$$=-\sin x-\int_0^x f(t)\,dt$$

$$f''(x)=-\cos x-f(x)$$

$$\Rightarrow f''(x)+f(x)=-\cos x$$

5. (c) : $$f(x)=\int_1^x \frac{\tan^{-1}t}{t}\,dt$$

$$\Rightarrow f\left(\frac{1}{x}\right)=\int_1^{1/x}\frac{\tan^{-1}t}{t}\,dt=-\int_1^x\frac{\cot^{-1}t}{t}\,dt$$

$$\Rightarrow f(x)-f\left(\frac{1}{x}\right)=\frac{\pi}{2}\log x$$

$$\Rightarrow f(e^2)-f\left(\frac{1}{e^2}\right)=\frac{\pi}{2}\log_e e^2=\frac{\pi}{2}\times 2=\pi.$$

6. (a) : $$\left(\frac{x}{1-x^2}\right)\times\frac{1}{x^3}=\left(\frac{1}{1-x^2}\right)\frac{1}{x^2}>1$$

So $$\tan^{-1}\left(\frac{x}{1-x^2}\right)+\tan^{-1}\left(\frac{1}{x^3}\right)$$

$$=\pi+\tan^{-1}\left(\frac{\frac{x}{1-x^2}+\frac{1}{x^3}}{1-\frac{1}{x^2(1-x^2)}}\right)$$

$$=\pi+\tan^{-1}\left(\frac{x^4+1-x^2}{(x^2-x^4-1)x}\right)$$

$$=\pi+\tan^{-1}\left(-\frac{1}{x}\right)=\frac{3\pi}{4}$$

$\Rightarrow x = 1.$

7. (a) : Any point on the line $x + y = 2a$ is $(t, 2a - t)$

Now equation of chord of contact is

$xt + y(2a - t) = a^2$

or $(x-y)t+2ay-a^2=0$

This passes through $F\left(\frac{a}{2},\frac{a}{2}\right)$, now equation of chord with F as mid point is $\frac{xa}{2}+\frac{ya}{2}=\frac{a^2}{2}$

$\Rightarrow \quad x+y=a$

Clearly this is parallel to the line $x + y = 2a$.

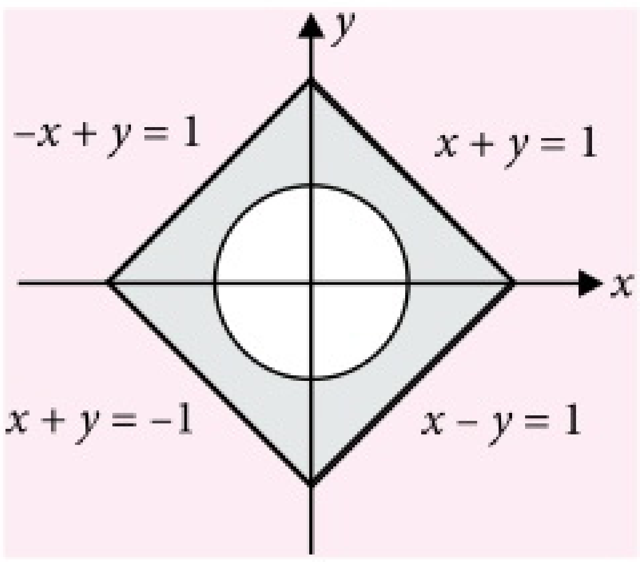
8. (b) : Both together form a square of side $\sqrt{2}$ units

$3x^2 + 3y^2 = 1$ is a circle of radius $\frac{1}{\sqrt{3}}$ units.

Area of circle = $\frac{\pi}{3}$ sq. units.

Area of square = 2 sq. units.

Required area = $\left(2-\frac{\pi}{3}\right)$ sq. units



9. (a) : $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{(1+\ln xy)^2}$

$\Rightarrow \quad xdy+ydx=\frac{xdx}{(1+\ln(xy))^2}$

$\Rightarrow \quad (1+\ln xy)^2 d(xy)=xdx$

Integrating both sides, we get

$\int(1+\ln t)^2 dt=\frac{x^2}{2}+C$ (Putting $t = xy$)

$\Rightarrow \quad (1+\ln t)^2 t-2\int(1+\ln t)\,dt=\frac{x^2}{2}+C$

$\Rightarrow \quad t(1+\ln t)^2-2t-2(t\ln t-t)=\frac{x^2}{2}+C$

$\Rightarrow \quad t(1+\ln t)^2-2t\ln t=\frac{x^2}{2}+C$

$\Rightarrow \quad t(1+(\ln t)^2)=\frac{x^2}{2}+C$

$\Rightarrow \quad xy(1+(\ln xy)^2)=\frac{x^2}{2}+C$.

10. (a) : Equation of PQ is $yt = x + t^2$

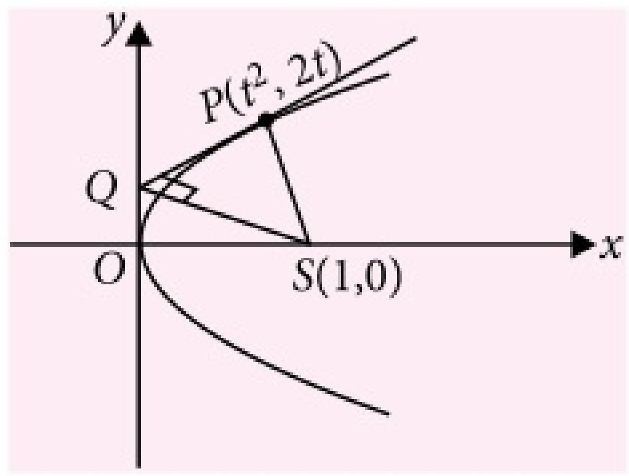
$Q\equiv(0,t) \Rightarrow PQ=\sqrt{t^4+t^2}=t\sqrt{1+t^2}$

$QS=\sqrt{1+t^2}$

$\Rightarrow \quad \Delta PQS=\frac{1}{2}PQ\times QS=\frac{1}{2}t(1+t^2)$

Which is increasing function of t

$(\Delta PQS)_{max.} = 1$ sq. unit.



YOU ASKED WE ANSWERED

Do you have a question that you just can't get answered?

Use the vast expertise of our mtg team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

Q1. If $\vec{m}$ and $\vec{n}$ are two perpendicular unit vectors and $\vec{a}$ is any vector, then show that

$$\vec{a} = (\vec{a}\cdot\vec{m})\vec{m} + (\vec{a}\cdot\vec{n})\vec{n} + [\vec{a}\;\vec{m}\;\vec{n}](\vec{m}\times\vec{n})$$

- Pratyush Kr. Singh, Patna

Ans. Since $\vec{m}$, $\vec{n}$ and $\vec{m}\times\vec{n}$ are non-coplanar vectors, therefore any vector $\vec{a}$ can be expressed as

$$\vec{a} = x\vec{m} + y\vec{n} + z(\vec{m}\times\vec{n}) \qquad \text{...(i)}$$

Taking dot product with $\vec{m}$ in (i), we have

$$\vec{a}\cdot\vec{m} = x|\vec{m}|^2 + y(\vec{m}\cdot\vec{n}) + [\vec{m}\;\vec{n}\;\vec{m}]$$
$$= x + 0 + 0 \quad [\vec{m}\perp\vec{n} \text{ and } |\vec{m}|^2 = 1]$$

$\Rightarrow \quad x = \vec{a}\cdot\vec{m}$

Taking dot product with $\vec{n}$ in (i), we have

$$\vec{a}\cdot\vec{n} = x(\vec{m}\cdot\vec{n}) + y|\vec{n}|^2 + [\vec{m}\;\vec{n}\;\vec{n}]$$
$$= 0 + y + 0 \quad [\vec{m}\perp\vec{n} \text{ and } |\vec{n}|^2 = 1]$$

$\Rightarrow \quad y = \vec{a}\cdot\vec{n}$

Taking dot product with $\vec{m}\times\vec{n}$ in (i), we have

$$\vec{a}\cdot(\vec{m}\times\vec{n}) = x[\vec{m}\;\vec{m}\;\vec{n}] + y[\vec{n}\;\vec{m}\;\vec{n}] + z|\vec{m}\times\vec{n}|^2$$
$$= 0 + 0 + z \qquad [|\vec{m}\times\vec{n}|^2 = 1]$$

$\Rightarrow \quad z = \vec{a}\cdot(\vec{m}\times\vec{n}) = [\vec{a}\;\vec{m}\;\vec{n}]$

Hence, $\vec{a} = (\vec{a}\cdot\vec{m})\vec{m} + (\vec{a}\cdot\vec{n})\vec{n} + [\vec{a}\;\vec{m}\;\vec{n}](\vec{m}\times\vec{n})$

Q2. The first 12 letters of English alphabet are written in a row at random. Find the probability that there are exactly four letters in between A and B.

- Biranchi Pattnaik, Cuttack

Ans. A and B can be arranged in ${}^{12}P_2 = 11 \times 12$ ways. Since we want 4 letters in between A and B, the order of the four letters appearing has no importance. A and B can take the following places.

Place for A	Place for B
1	6
2	7
3	8
4	9
5	10
6	11
7	12

A and B can be interchanged. Therefore required probability is $\dfrac{14}{11\times 12} = \dfrac{7}{66}$

Q3. Let $f(x) = x^3 + 2x^2 + x + 5$. Show that $f(x) = 0$ has only one real root α such that $[\alpha] = -3$.

- Sujata Sahoo, Warangal

Ans. We have

$$f(x) = x^3 + 2x^2 + x + 5,\ x \in R$$

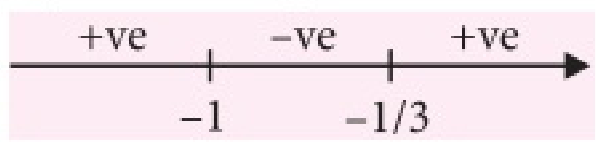
and $f'(x) = 3x^2 + 4x + 1 = (x+1)(3x+1),\ x \in R$

Drawing the number line for $f'(x)$, we have

$f(x)$ strictly increasing in $(-\infty, -1)$

strictly decreasing in $(-1, -1/3)$

strictly increasing in $(-1/3, \infty)$



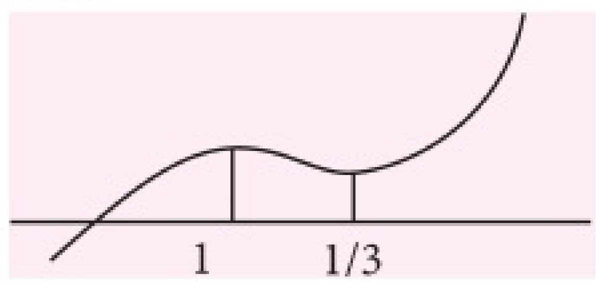
Also, we have

$$f(-1) = -1 + 2 - 1 + 5 = 5$$

and $f\left(\dfrac{-1}{3}\right) = \dfrac{-1}{27} + \dfrac{2}{9} - \dfrac{1}{3} + 5 = 5 - \dfrac{4}{27} \approx 4.85$

The graph of $f(x)$ shows that $f(x)$ cut the X-axis only once.

Now, we have



$$f(-3) = -27 + 12 - 3 + 5 = -13$$

and $f(-2) = -8 + 8 - 2 + 5 = 3$

which are of opposite signs. This proves that the curve cuts the X-axis somewhere between -2 and -3.

$\Rightarrow f(x) = 0$ has a root α lying between -2 and -3.

Hence, $[\alpha] = -3$.

ENRICH YOUR CONCEPTS ON XI

Limits and Derivatives & Mathematical Reasoning

Based on NCERT Pattern

Series-7

LIMITS

Left Hand Limit	The left hand limit of f at $x = a$ is the expected value of f at $x = a$ given the values of f near x to the left of a. It is denoted by $\lim_{x\to a^-} f(x)$.
Right Hand Limit	The right hand limit of f at $x = a$ is the expected value of f at $x = a$ given the values of f near x to the right of a. It is denoted by $\lim_{x\to a^+} f(x)$.

LIMIT OF A FUNCTION AT A POINT

Limit of a function at $x = a$ is the value of $f(x)$ at $x = a$ *i.e.*, $\lim_{x\to a} f(x)$

Note : $\lim_{x\to a} f(x)$ exists at $x = a$ iff

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

ALGEBRA OF LIMITS

Let $f(x)$ and $g(x)$ be two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then,

(i) $\lim_{x\to a}\left[f(x)+g(x)\right] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a}\left[f(x)-g(x)\right] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\left[f(x)\cdot g(x)\right] = \lim_{x\to a} f(x)\cdot \lim_{x\to a} g(x)$

(iv) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$

(v) $\lim_{x\to a}\left\{f(x)\right\}^{\{g(x)\}} = \left\{\lim_{x\to a} f(x)\right\}^{\lim_{x\to a} g(x)}$

(vi) $\lim_{x\to a} \lambda\cdot f(x) = \lambda\cdot \lim_{x\to a} f(x)$, λ being some real number.

LIMITS OF POLYNOMIALS AND RATIONAL FUNCTIONS

- A function $f(x)$ is said to be a polynomial function if $f(x) = a_0 + a_1x + a_2x_2 + \ldots\ldots + a_n x^n$, $0 \le i \le$ n where a_i's are real numbers. If $a_i = 0\ \forall$ $0 \le$ i $\le n$, then $f(x)$ is called zero function.
  Then $\lim_{x\to a} f(x) = f(a)$
- A function f is said to be a rational function, if $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials such that $h(x) \neq 0$. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a}\frac{g(x)}{h(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} h(x)} = \frac{g(a)}{h(a)}$$

SOME IMPORTANT PROPERTIES

(i) If f and g be two real valued functions such that $f(x) \le g(x)$ for all x lies in the common domain of f and g, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$, when both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist at point a.

(ii) (Sandwich theorem) : Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all $x \in \{\text{dom}\, f(x) \cap \text{dom}\, g(x) \cap \text{dom}\, h(x)\}$. Now, for $a \in R$ if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ exist and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$, then $\lim_{x\to a} g(x) = l$

SOME STANDARD LIMITS

(i) $\lim_{x\to a}\dfrac{x^n - a^n}{x-a} = na^{n-1}$, n being a positive integer

(ii) $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$

(iii) $\lim_{x\to a}\frac{\sin(x-a)}{x-a}=1$

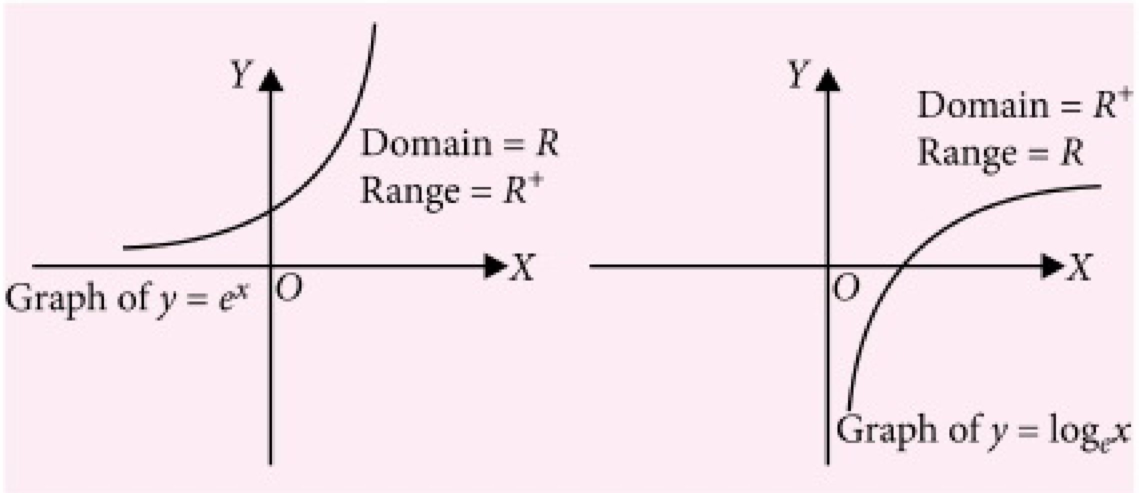
(iv) $\lim_{x\to 0}\cos x=1$

(v) $\lim_{x\to 0}\frac{\tan x}{x}=1$

(vi) $\lim_{x\to a}\frac{\tan(x-a)}{x-a}=1$

(vii) $\lim_{x\to 0}\frac{1-\cos x}{x}=0$

LIMITS INVOLVING EXPONENTIAL AND LOGARITHMIC FUNCTIONS



Results

(i) $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$

(ii) $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e a,\ a>1$

(iii) $\lim_{x\to 0}\frac{e^x-1}{x}=1$

(iv) $\lim_{x\to 0}(1+x)^{1/x}=e$

DERIVATIVES

Definition : Let $f(x)$ be a real valued function and a be any point in its domain of definition. The derivative of $f(x)$ at $x = a$ is denoted by $f'(a)$ and defined as $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$, provided this limit exists.

FIRST PRINCIPLE OF DERIVATIVE

- Let $f(x)$ be a function finitely differentiable at every point on its domain. Then, its first principle of derivative is given by

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

ALGEBRA OF DERIVATIVE OF FUNCTIONS

- Let $f(x)$ and $g(x)$ be two functions such that their derivatives are defined. Then

(i) $\frac{d}{dx}[f(x)+g(x)]=\frac{d}{dx}f(x)+\frac{d}{dx}g(x)$

(ii) $\frac{d}{dx}[f(x)-g(x)]=\frac{d}{dx}f(x)-\frac{d}{dx}g(x)$

(iii) $\frac{d}{dx}[f(x)\cdot g(x)]=f(x)\frac{d}{dx}g(x)+g(x)\frac{d}{dx}f(x)$

It is known as Product rule.

(iv) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{g(x)\frac{d}{dx}f(x)-f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$, provided $g(x)\neq 0$

It is known as Quotient Rule.

DERIVATIVE OF POLYNOMIAL FUNCTION

- Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0$ be a polynomial function of degree n in x where $a_i \in R$, $i = 0, 1, 2, ..., n$ and $a_n \neq 0$. Then

$$\frac{d}{dx}f(x)=na_n x^{n-1}+(n-1)a_{n-1}x^{n-2}+(n-2)a_{n-2}x^{n-3} +.........+ 2a_2 x + a_1$$

DERIVATIVES OF SOME STANDARD FUNCTIONS

(i) $\frac{d}{dx}(c)=0$, c is independent of x.

(ii) $\frac{d}{dx}(x^n)=nx^{n-1}$

(iii) $\frac{d}{dx}(\sin x)=\cos x$

(iv) $\frac{d}{dx}(\cos x)=-\sin x$

(v) $\frac{d}{dx}(\tan x)=\sec^2 x,\ \left\{x\neq n\pi+\frac{\pi}{2}\right\}$

(vi) $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x,\ \{x\neq n\pi\}$

(vii) $\frac{d}{dx}(\sec x)=\sec x\tan x,\ \left\{x\neq n\pi+\frac{\pi}{2}\right\}$

(viii) $\frac{d}{dx}(\text{cosec}\,x)=-\text{cosec}\,x\cot x,\ \{x\neq n\pi\}$

(ix) $\frac{d}{dx}(a^x)=a^x\log_e a$

(x) $\dfrac{d}{dx}(e^x) = e^x$

(xi) $\dfrac{d}{dx}(\log x) = \dfrac{1}{x}, (x > 0)$

(xii) $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \log_e a}$

(xiii) $\dfrac{d}{dx}\{c.f(x)\} = c\dfrac{d}{dx}(f(x))$, c being a constant.

MATHEMATICAL REASONING

STATEMENT

- If a sentence is either true or false but not both, then it is called a statement.

NEGATION OF A STATEMENT

- If p is a statement, then the negation of p is also a statement and is denoted by $\sim p$, and read as 'not p'.

SIMPLE STATEMENT

- A statement, if cannot be broken into two or more sentences, is a simple statement. Generally, small letters p, q, r,....... denote simple statements.

COMPOUND STATEMENT

1. A statement formed by two or more simple statements by using words "and", "or", etc. is called a compound statement. In this case, each statement is called component statement.
2. The compound statement with "AND" is

(i) True if all its component statements are true.

(ii) False if any of its component statements is false.

3. The compound statement with "OR" is

(i) True when one component statement is true or both the component statements are true.

(ii) False when both the component statements are false.

4. A statement is said to be valid or invalid according as it is true or false.

QUANTIFIERS

- Many mathematical statements contain phrases 'there exists' and 'for all' or 'for every'. These phrases are called quantifiers.

IMPLICATIONS

- If any two simple statements can be combined by the word 'if-then', then it is called the implication and it is denoted by the symbol '$\Rightarrow$' or '$\rightarrow$'.
- **If then implication :** A sentence "if p then q" can be written in the following ways :

(i) p implies q (denoted by $p \Rightarrow q$)

(ii) p is sufficient condition for q

(iii) q is necessary condition for p

(iv) p only if q

(v) $\sim q$ implies $\sim p$

CONTRAPOSITIVE

- If p and q are two statements, then the contrapositive of the statement "If p then q" is "if $\sim q$ then $\sim p$".

CONVERSE

- If p and q are two statements, then the converse of the implication "if p then q" is "if q then p".

INVERSE

- If p and q are two statements, then the inverse of "if p then q" is "if $\sim p$ then $\sim q$."

If and only if Implication :

- If p and q are two statements, then the compound statement $p \Rightarrow q$ and $q \Rightarrow p$ is called if and only if implication and is denoted by $p \Leftrightarrow q$.
- If p and q are two mathematical statements, then the statement :

(i) "If p then q" is true if

(a) p is true $\Rightarrow$ q is true

or

(b) q is false $\Rightarrow$ p is false

or

(c) p is true and q is false leads us to a contradiction.

(ii) "p if and only if q" is true, if

(a) p is true $\Rightarrow$ q is true and

(b) q is true $\Rightarrow$ p is true.

VERY SHORT ANSWER TYPE (1 MARK)

1. Let $f(x) = \begin{cases} \cos x, \text{if } x \geq 0 \\ x + k, \text{if } x < 0 \end{cases}$. Find the value of constant k, given that $\lim\limits_{x \to 0} f(x)$ exists.

2. Evaluate $\lim_{x\to 0}\frac{\sin x^\circ}{x}$.

3. Differentiate $\sin x^3$ w.r.t. x.

4. Differentiate $5(2^{3\log_2 x})$ with respect to x.

5. Which of the following is a statement (or proposition)? Give reasons for your answer.

   (i) $x^2 + 5x + 6 = 0$.

   (ii) There is no rain without clouds.

6. Evaluate $\lim_{x\to -\infty}\frac{\sqrt{x^2+1}}{x}$.

7. Write down the negation of each of the following statements.

   (i) Deepak is smart and healthy.

   (ii) $2 + 4 > 5$ or $3 + 4 < 6$.

   (iii) $x = 2 \Rightarrow x^2 = 4$.

   (iv) ΔABC is isosceles, if and only if $\angle B = \angle C$

SHORT ANSWER TYPE (4 MARKS)

8. Evaluate $\lim_{x\to 2}\frac{x^5-32}{x^3-8}$.

9. Find $\frac{dy}{dx}$ when $y=\frac{x^2\sin x}{1-x}$.

10. If $y=\sqrt{\frac{1-\cos 2x}{1+\cos 2x}},\ x\in\left(0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right)$, then find $\frac{dy}{dx}$.

11. Evaluate:

$$\lim_{x\to 0}\frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$$

12. Prove that : $\lim_{x\to\pi/4}\frac{\tan^3 x-\tan x}{\cos\left(x+\frac{\pi}{4}\right)}=-4$

13. Find $\frac{dy}{dx}$, when $y=\frac{(\sqrt{x+a}-\sqrt{x-a})^2}{\sqrt{x^2-a^2}}$, where $x > a > 0$.

14. Find $\frac{d}{dx}\left(\frac{\sin x}{x}\right)$, by using first principle of derivative.

15. Show that the statement

   p : If x is a real number such that $x^3 + 4x = 0$, then x is "0" is true by

   (i) direct method

   (ii) method of contradiction

   (iii) method of contrapositive

SOLUTIONS

1. We have, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$

$$\Rightarrow \lim_{x\to 0} x+k=\lim_{x\to 0}\cos x \Rightarrow 0+k=\cos 0$$

$\Rightarrow k = 1$

2. $$\lim_{x\to 0}\frac{\sin x^\circ}{x}=\lim_{x\to 0}\frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$=\frac{\pi}{180}\lim_{\frac{\pi x}{180}\to 0}\left(\frac{\sin\frac{\pi x}{180}}{\frac{\pi x}{180}}\right)=\frac{\pi}{180}(1)=\frac{\pi}{180}$$

3. Let $y = \sin x^3$ ...(i)

   Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx}=\cos x^3\frac{d}{dx}(x^3)=(\cos x^3)\times 3x^2=3x^2\cos x^3.$$

4. $$\frac{d}{dx}(5\cdot 2^{3\log_2 x})=5\cdot\frac{d}{dx}(2^{3\log_2 x})$$

$$=5\cdot\frac{d}{dx}(2^{\log_2 x^3})=5\cdot\frac{d}{dx}(x^3)=15x^2.$$

$[\because\ a^{\log_a n}=n]$

5. **(i)** $x^2 + 5x + 6 = 0$ is not a statement, because its truth or falsity cannot be confirmed without knowing the value of x.

(ii) It is scientifically established natural phenomenon that cloud is formed before it rains. Therefore, this sentence is always true. Hence, it is a statement.

6. $$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x}=\lim_{x\to-\infty}\frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x}$$

$$=\frac{5\times 2^4}{3\times 2^2}=\frac{5}{3}\times 2^2=\frac{20}{3}$$

(as $x\to-\infty$, $\therefore x<0\Rightarrow |x|=-x$)

$$= \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to-\infty} -\sqrt{1+\frac{1}{x^2}} = -1$$

$$\left(\text{As } x \to -\infty, \frac{1}{x} \to 0^- \Rightarrow \lim_{x\to\infty} \frac{1}{x} = 0\right)$$

7. **(i)** Deepak is not smart or he is not healthy.

(ii) $2 + 4 \le 5$ and $3 + 4 \ge 6$

(iii) $x = 2$ and $x^2 \ne 4$

(iv) Either ΔABC is isosceles and $\angle B \ne \angle C$ or $\angle B = \angle C$ and ΔABC is not isosceles.

8. Now, $\lim_{x\to2} \frac{x^5-32}{x^3-8} = \lim_{x\to2} \frac{x^5-2^5}{x^3-2^3}$

$$= \lim_{x\to2} \frac{\frac{x^5-2^5}{x-2}}{\frac{x^3-2^3}{x-2}} = \frac{(5\times2^{5-1})}{3\times2^{3-1}}$$

$$= \frac{5\times2^4}{3\times2^2} = \frac{5}{3}\times2^2 = \frac{20}{3} \quad \left[\because \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

9. Given, $y = \frac{x^2\sin x}{1-x}$ ...(i)

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1-x)\frac{d}{dx}(x^2\sin x) - (x^2\sin x)\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$= \frac{(1-x)\left\{x^2\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(x^2)\right\} - x^2\sin x(0-1)}{(1-x)^2}$$

$$= \frac{(1-x)\{x^2\cos x + (\sin x)(2x)\} + x^2\sin x}{(1-x)^2}$$

$$= \frac{(1-x)x^2\cos x + 2x(1-x)\sin x + x^2\sin x}{(1-x)^2}$$

$$= \frac{(1-x)x^2\cos x + x(2-x)\sin x}{(1-x)^2}, x \ne 1.$$

10. We have, $y = \sqrt{\frac{1-\cos2x}{1+\cos2x}} = \sqrt{\frac{2\sin^2x}{2\cos^2x}} = \sqrt{\tan^2x}$

$$\Rightarrow \quad y = |\tan x|, \text{ where } x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow y = \begin{cases} \tan x, x \in \left(0, \frac{\pi}{2}\right) \\ -\tan x, x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} \sec^2 x, \text{ if } x \in \left(0, \frac{\pi}{2}\right) \\ -\sec^2 x, \text{ if } x \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$$

11. We have,

$$\lim_{x\to0} \frac{8}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)$$

$$= \lim_{x\to0} \frac{8}{x^8}\left\{\left(1-\cos\frac{x^2}{2}\right)-\cos\frac{x^2}{4}\left(1-\cos\frac{x^2}{2}\right)\right\}$$

$$= \lim_{x\to0} 8\left\{\frac{1-\cos\frac{x^2}{2}}{x^4}\right\}\left\{\frac{1-\cos\frac{x^2}{4}}{x^4}\right\}$$

$$= \lim_{x\to0} 32\times\left\{\frac{\sin\frac{x^2}{4}}{x^2}\right\}^2\times\left\{\frac{\sin\frac{x^2}{8}}{x^2}\right\}^2$$

$$= 32\lim_{x\to0}\left\{\frac{\sin\frac{x^2}{4}}{4\left(\frac{x^2}{4}\right)}\right\}^2\times\lim_{x\to0}\left\{\frac{\sin\frac{x^2}{8}}{8\left(\frac{x^2}{8}\right)}\right\}^2$$

$$= 32\times\left(\frac{1}{4}\right)^2\times\left(\frac{1}{8}\right)^2 = 32\times\frac{1}{16}\times\frac{1}{64} = \frac{1}{32}$$

12. We have, L.H.S. $\lim_{x\to\pi/4} \frac{\tan^3x-\tan x}{\cos(x+(\pi/4))}$

$$= \lim_{x\to\pi/4} \frac{\tan x(\tan x-1)(\tan x+1)}{\cos(x+(\pi/4))}$$

$$= \lim_{x\to\pi/4} \frac{\tan x(\sin x-\cos x)(\tan x+1)}{\cos x\cos(x+(\pi/4))}$$

$$= -\sqrt{2}\lim_{x\to\pi/4} \frac{\tan x\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)(\tan x+1)}{\cos x\cos(x+(\pi/4))}$$

$$= -\sqrt{2}\lim_{x\to\pi/4}\frac{\tan x\cos(x+(\pi/4)).(\tan x+1)}{\cos x\cos(x+(\pi/4))}$$

$$= -\sqrt{2}\times\lim_{x\to\pi/4}\frac{\tan x(\tan x+1)}{\cos x}$$

$$= -\sqrt{2}\times 2\sqrt{2} = -4 = \text{R.H.S.}$$

13. Given that $y = \dfrac{(\sqrt{x+a}-\sqrt{x-a})^2}{\sqrt{x^2-a^2}}$

$$= \frac{x+a+x-a-2\sqrt{x+a}\sqrt{x-a}}{\sqrt{x^2-a^2}}$$

$$\Rightarrow \quad y = \frac{2x-2\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}} = \frac{2x}{\sqrt{x^2-a^2}} - \frac{2\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}}$$

or $\quad y = \dfrac{2x}{\sqrt{x^2-a^2}} - 2 \qquad \text{...(i)}$

Differentiating (i) w.r.t. x, we get

$$\frac{dy}{dx} = 2\frac{d}{dx}\left(\frac{x}{\sqrt{x^2-a^2}}\right) - \frac{d}{dx}(2)$$

$$= 2\left\{\frac{\sqrt{x^2-a^2}\frac{d}{dx}(x) - x\frac{d}{dx}(\sqrt{x^2-a^2})}{(\sqrt{x^2-a^2})^2}\right\} - 0$$

$$= 2\left\{\frac{\sqrt{x^2-a^2}\times 1 - x\times\frac{1}{2}(x^2-a^2)^{-1/2}\frac{d}{dx}(x^2-a^2)}{x^2-a^2}\right\}$$

$$= 2\left\{\frac{x^2-a^2-x^2}{\sqrt{x^2-a^2}(x^2-a^2)}\right\} = \frac{-2a^2}{(x^2-a^2)^{3/2}}.$$

14. Let $f(x) = \dfrac{\sin x}{x}$, then $f(x+\Delta x) = \dfrac{\sin(x+\Delta x)}{x+\Delta x}$

Hence $\qquad f'(x) = \lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$

$$= \lim_{\Delta x\to 0}\frac{\frac{\sin(x+\Delta x)}{x+\Delta x} - \frac{\sin x}{x}}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{x\sin(x+\Delta x)-(x+\Delta x)\sin x}{x(x+\Delta x)\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{x\{\sin(x+\Delta x)-\sin x\} - \Delta x\sin x}{x(x+\Delta x)\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{(x+\Delta x)\Delta x} - \lim_{\Delta x\to 0}\frac{\sin x}{x(x+\Delta x)}$$

$$= \left[\lim_{\Delta x\to 0}\frac{2\cos\left(\frac{x+\Delta x+x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{(x+\Delta x)\Delta x}\right] - \frac{\sin x}{x^2}$$

$$= \lim_{\Delta x\to 0}\frac{\cos\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{(x+\Delta x)\frac{\Delta x}{2}} - \frac{\sin x}{x^2}$$

$$= \left(\frac{1}{x}\cos x\right)\times 1 - \frac{\sin x}{x^2} = \frac{x\cos x - \sin x}{x^2}$$

$$\left(\because \text{when } \Delta x\to 0 \Rightarrow \frac{\Delta x}{2}\to 0 \ \therefore \ \underset{\Delta x\to 0}{\text{Lt}}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1\right)$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}.$$

15. Let q and r be the statements given by

q : x is a real number such that $x^3 + 4x = 0$

r : x is 0.

Then, p : If q, then r.

(i) Direct method : Let q be true. Then,

x is a real number such that $x^3 + 4x = 0$

$\Rightarrow$ x is a real number such that $x(x^2 + 4) = 0$

$\Rightarrow$ $x = 0 \qquad [\because \ x \in R \therefore x^2 + 4 \neq 0]$

$\Rightarrow$ r is true.

Thus, q is true $\Rightarrow$ r is true.

Hence, p is true.

(ii) Method of contradiction : If possible, let p be not true. Then,

p is not true.

$\Rightarrow$ $\sim p$ is true

$\Rightarrow$ $\sim(q \Rightarrow r)$ is true $\qquad [\because \ p : q \Rightarrow r]$

$\Rightarrow$ q and $\sim r$ is true $\qquad [\because \sim(q \Rightarrow r) \cong q \text{ and } \sim r]$

$\Rightarrow$ x is a real number such that $x^3 + 4x = 0$ and $x \neq 0$

$\Rightarrow$ $x = 0$ and $x \neq 0$

This is a contradiction.

Hence, p is true.

(iii) Method of contrapositive : Let r be not true.

Then, q is not true.

$\Rightarrow$ $x \neq 0, x \in R$

$\Rightarrow$ $x(x^2 + 4) \neq 0, x \in R$

$\Rightarrow$ q is not true

Thus, $\sim r \Rightarrow \sim q$

Hence, p is true.

OLYMPIAD CORNER

CHALLENGING PROBLEMS FOR RMO, InMO, IMO & OTHER CONTESTS

1. Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that
$$\frac{1}{O_1A_1^2-r_1^2}+\frac{1}{O_2A_2^2-r_2^2}+\frac{1}{O_3A_3^2-r_3^2}+\frac{1}{O_4A_4^2-r_4^2}=0.$$

2. Determine the greatest positive integer k that satisfies the following property. The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots\ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots\ldots, k)$ there exist two distinct elements of A_i whose sum is n.

3. Consider a polynomial $P(x) = (x + d_1)(x + d_2)\ldots\ldots(x + d_9)$, where $d_1, d_2, \ldots\ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$, the number $P(x)$ is divisible by a prime number greater than 20.

4. Let p be an odd prime number. For every integer a, define the number
$$S_a=\frac{a}{1}+\frac{a^2}{2}+\ldots\ldots+\frac{a^{p-1}}{p-1}.$$
Let m and n be integers such that
$$S_3+S_4-3S_2=\frac{m}{n}.$$
Prove that p divides m.

5. Let a, b and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2=3$. Prove that
$$\frac{a}{(b+c-a)^2}+\frac{b}{(c+a-b)^2}+\frac{c}{(a+b-c)^2}\geq\frac{3}{(abc)^2}. \quad \ldots(1)$$

SOLUTIONS

1. Let M be the point of intersection of the diagonals A_1A_3 and A_2A_4. On each diagonal choose a direction and let x, y, z, and w be the signed distances from M to the points A_1, A_2, A_3 and A_4, respectively.

Let w_1 be the circumcircle of the triangle $A_2A_3A_4$ and let B_1 be the second intersection point of w_1 and A_1A_3 (thus, $B_1 = A_3$ if and only if A_1A_3 is tangent to w_1). Since the expression $O_1A_1^2-r_1^2$ is the power of the point A_1 with respect to w, we get
$$O_1A_1^2-r_1^2=A_1B_1\cdot A_1A_3.$$
On the other hand, from the equality $MB_1 \cdot MA_3 = MA_2 \cdot MA_4$, we obtain $MB_1 = yw/z$.
Hence, we have
$$O_1A_1^2-r_1^2=\left(\frac{yw}{z}-x\right)(z-x)=\frac{z-x}{z}(yw-xz).$$
Substituting the analogous expressions into the sought sum, we get
$$\sum_{i=1}^{4}\frac{1}{O_iA_i^2-r_i^2}$$
$$=\frac{1}{yw-xz}\left(\frac{z}{z-x}-\frac{w}{w-y}+\frac{x}{x-z}-\frac{y}{y-w}\right)=0,$$
as desired.

2. There are various examples showing that $k = 3$ does indeed have the property under consideration. *e.g.*, one can take
$$A_1 = \{1, 2, 3\} \cup \{3m | m \geq 4\},$$

$A_2 = \{4, 5, 6\} \cup \{3m - 1 | m \ge 4\}$,

$A_3 = \{7, 8, 9\} \cup \{3m - 2 | m \ge 4\}$,

To check that this partition fits, we notice first that the sums of two distinct elements of A_i obviously represent all numbers $n \ge 1 + 12 = 13$ for $i = 1$, all numbers $n \ge 4 + 11 = 15$ for $i = 2$, and all numbers $n \ge 7 + 10 = 17$ for $i = 3$. So, we are left to find representations of the numbers 15 and 16 as sums of two distinct elements of A_3. These are $15 = 7 + 8$ and $16 = 7 + 9$.

Let us now suppose that for some $k \ge 4$, there exist sets $A_1, A_2, \ldots, A_k$ satisfying the given property. Obviously, the sets $A_1, A_2, A_3, A_4 \cup \ldots \cup A_k$ also satisfy the same property, so one may assume $k = 4$.

Put $B_i = A_i \cap \{1, 2, \ldots, 23\}$ for $i = 1, 2, 3, 4$. Now for any index i each of the ten numbers $15, 16, \ldots, 24$ can be written as sum of two distinct elements of B_i. Therefore this set needs to contain at least five elements. As we also have $|B_1| + |B_2| + |B_3| + |B_4| = 23$, there has to be some index j for which $|B_j| = 5$. Let $B_j = \{x_1, x_2, x_3, x_4, x_5\}$. Finally, now the sums of two distinct elements of A_j representing the numbers $15, 16, \ldots, 24$ should be exactly all the pairwise sums of the elements of B_j. Calculating the sum of these numbers in two different ways, we reach

$4(x_1 + x_2 + x_3 + x_4 + x_5) = 15 + 16 + \ldots + 24 = 195$.

Thus the number 195 should be divisible by 4, which is false. This contradiction completes our solution.

3. Note that the statement of the problem is invariant under translations of x; hence without loss of generality we may suppose that the numbers $d_1, d_2, \ldots, d_9$ are positive.

The key observation is that there are only eight primes below 20, while $P(x)$ involves more than eight factors.

We shall prove that $N = d^8$ satisfies the desired property, where $d = \max\{d_1, d_2, \ldots, d_9\}$. Suppose for the sake of contradiction that there is some integer $x \ge N$ such that $P(x)$ is composed of primes below 20 only. Then for every index $i \in \{1, 2, \ldots, 9\}$ the number $x + d_i$ can be expressed as product of powers of the first 8 primes.

Since $x + d_i > x \ge d^8$ there is some prime power $f_i > d$ that divides $x + d_i$. Invoking the pigeonhole principle we see that there are two distinct indices i and j such that f_i and f_j are powers of the same prime number. For reasons of symmetry, we may suppose that $f_i \le f_j$. Now both of the numbers $x + d_i$ and $x + d_j$ are divisible by f_i and hence so is their difference $d_i - d_j$. But as $0 < |d_i - d_j| \le \max(d_i, d_j) \le d < f_i$, this is impossible. Thereby the problem is solved.

4. For rational numbers p_1/q_1 and p_2/q_2 with the denominators q_1, q_2 not divisible by p, we write $p_1/q_1 \equiv p_2/q_2 \pmod p$ if the numerator $p_1q_2 - p_2q_1$ of their difference is divisible by p.

We start with finding an explicit formula for the residue of S_a modulo p. Note first that for every $k = 1, \ldots, (p - 1)$ the number $\binom{p}{k}$ is divisible by p, and

$$\frac{1}{p}\binom{p}{k} = \frac{(p-1)(p-2)\ldots(p-k+1)}{k!} \equiv \frac{(-1)\cdot(-2)\ldots(-k+1)}{k!}$$

$$= \frac{(-1)^{k-1}}{k} \pmod p$$

Therefore, we have

$$S_a = -\sum_{k=1}^{p-1} \frac{(-a)^k(-1)^{k-1}}{k} \equiv -\sum_{k=1}^{p-1} (-a)^k \cdot \frac{1}{p}\binom{p}{k} \pmod p.$$

The number on the right-hand side is integer. Using the binomial formula, we express it as

$$-\sum_{k=1}^{p-1} (-a)^k \cdot \frac{1}{p}\binom{p}{k}$$

$$= -\frac{1}{p}\left(-1 - (-a)^p + \sum_{k=0}^{p} (-a)^k \binom{p}{k}\right)$$

$$= \frac{(a-1)^p - a^p + 1}{p}$$

Since p is odd. So, we have

$$S_a \equiv \frac{(a-1)^p - a^p + 1}{p} (\text{mod } p).$$

Finally, using the obtained formula, we get

$$S_3 + S_4 - 3S_2 = \frac{(2^p - 3^p + 1) + (3^p - 4^p + 1) - 3(1^p - 2^p + 1)}{p}$$

$$= \frac{4.2^p - 4^p - 4}{p} = -\frac{(2^p - 2)^2}{p} (\text{mod } p).$$

By FERMAT'S theorem, $p|2^p - 2$,
so $p^2|(2^p - 2)^2$ and
hence $S_3 + S_4 - 3S_2 \equiv 0 \pmod p$.

5. The condition $b + c > \sqrt{2}$ implies $b^2 + c^2 > 1$, so $a^2 = 3 - (b^2 + c^2) < 2$, *i.e.*, $a < \sqrt{2} < b + c$. Hence we have $b + c - a > 0$, and also $c + a - b > 0$ and a $+ b -$ c > 0 for similar reasons.

We will use the variant of HOLDER's inequality

$$\frac{x_1^{p+1}}{y_1^p} + \frac{x_1^{p+1}}{y_1^p} + \ldots + \frac{x_n^{p+1}}{y_n^p} \geq \frac{(x_1 + x_2 + \ldots.. + x_n)^{p+1}}{(y_1 + y_2 + \ldots.. + y_n)^p}.$$

Which holds for all positive real numbers p, x_1, x_2,....., x_n, y_1, y_2,......, y_n. Applying it to the left-hand side of (1) with $p = 2$ and $n = 3$, we get

$$\sum \frac{a}{(b+c-a)^2} = \sum \frac{(a^2)^3}{a^5(b+c-a)^2}$$

$$\geq \frac{(a^2+b^2+c^2)^3}{\left(\sum a^{5/2}(b+c-a)\right)^2}$$

$$= \frac{27}{\left(\sum a^{5/2}(b+c-a)\right)^2} \qquad \ldots(2)$$

To estimate the denominator of the right-hand part, we use an instance of SCHUR's inequality, namely

$$\sum a^{3/2}(a-b)(a-c) \geq 0,$$

which can be rewritten as

$$\sum a^{5/2}(b+c-a) \leq abc(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Moreover, by the inequality between the arithmetic mean and the fourth power mean we also have

$$\left(\frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{3}\right)^4 \leq \frac{a^2+b^2+c^2}{3} = 1,$$

i.e., $\sqrt{a}+\sqrt{b}+\sqrt{c} \leq 3$. Hence, (2) yields

$$\sum \frac{a}{(b+c-a)^2} \geq \frac{27}{(abc(\sqrt{a}+\sqrt{b}+\sqrt{c}))^2} \geq \frac{3}{a^2b^2c^2}$$

MOCK TEST PAPER 2015

JEE Main

Exam on 4th April 2015

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1. A variable line through $A(6, 8)$ meets the curve $x^2 + y^2 = 2$ at B and C. P is a point on BC such that AB, AP, AC are in H.P. The minimum distance of the origin from the locus of P is
(a) 1 (b) 1/2
(c) 1/3 (d) 1/5

2. Number of solutions of the equation $4\sin^2 x + \tan^2 x + \cot^2 x + \text{cosec}^2 x = 6$ in $[0, \pi]$ is
(a) 0 (b) 2
(c) 8 (d) 4

3. If a line with direction ratios 2 : 2 : 1 intersects the line $\frac{x-7}{3} = \frac{y-5}{2} = \frac{z-3}{1}$ and $\frac{x-1}{2} = \frac{y+1}{4} = \frac{z+1}{3}$ at A and B, then AB =
(a) $\sqrt{2}$ units (b) 2 units
(c) $\sqrt{3}$ units (d) 3 units

4. Let $S \equiv \{a \in N, a \le 100\}$. If the equation $[\tan^2 x] - \tan x - a = 0$ has real roots, then number of elements in S is, [where [.] is greatest integer function]
(a) 10 (b) 8
(c) 9 (d) 0

5. The lengths of two opposite edges of a tetrahedron are a, b. Their shortest distance is d and the angle between them is θ. Then its volume is
(a) $\frac{1}{2}abd\sin\theta$ (b) $\frac{1}{3}abd\cos\theta$
(c) $\frac{1}{6}abd\sin\theta$ (d) $\frac{1}{6}abd\cos\theta$

6. If $\underset{x\to 0}{\text{Lt}} \frac{((a-n)nx - \tan x)\sin nx}{x^2} = 0$ where n is a non zero real number, then a is equal to
(a) 0 (b) $\frac{n+1}{n}$ (c) n (d) $n + \frac{1}{n}$

7. The solution curves of the differential equation
$$(xdx + ydy)\sqrt{x^2 + y^2} = (xdy - ydx)(\sqrt{1 - x^2 - y^2})$$
are
(a) circles of radius 1, passing through the origin
(b) circles of radius 1/2, passing through the origin
(c) circles not passing through origin
(d) solution curve is not a circle

8. The value of ${}^{2000}C_2 + {}^{2000}C_5 + {}^{2000}C_8 + \ldots + {}^{2000}C_{2000} = ?$
(a) $\frac{2^{1999}-1}{3}$ (b) $\frac{2^{1999}+1}{3}$
(c) $\frac{2^{2000}+1}{3}$ (d) $\frac{2^{2000}-1}{3}$

9. If $A = \int_1^{\sin\theta} \frac{t\,dt}{1+t^2} dt, B = \int_1^{\text{cosec}\theta} \frac{1}{t(1+t^2)} dt$

then $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2 + B^2 & -1 \end{vmatrix} = ?$
(a) $\sin\theta$ (b) $\text{cosec}\theta$
(c) 0 (d) 1

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10. The curves $C_1 : y = x^2 - 3$; $C_2 : y = kx^2$, $k < 1$ intersect each other at two different points. The tangent drawn to C_2, at one of the points of intersection $A = [(a, y_1); (a > 0)]$ meets C_1 again at $B = [(1, y_2); (y_1 \neq y_2)]$. Then value of a = ?

(a) 4 (b) 3 (c) 2 (d) 1

11. Let $A(1, 2)$, $B(3, 4)$ be two points and $C(x, y)$ be a point such that area of ΔABC is 3 sq.units and $(x - 1)(x - 3) + (y - 2)(y - 4) = 0$. Then maximum number of positions of C, in the xy plane is

(a) 2 (b) 4
(c) 8 (d) none of these

12. If $\sin^2(\theta - \alpha)\cos\alpha = \cos^2(\theta - \alpha)\sin\alpha = m\sin\alpha\cos\alpha$, then

(a) $|m| \leq \frac{1}{\sqrt{2}}$ (b) $|m| \geq \frac{1}{\sqrt{2}}$
(c) $|m| \geq 1$ (d) $|m| \leq 1$

13. If z is a complex number, then the number of complex numbers satisfying the equation $z^{2009} = \bar{z}$ is

(a) 3 (b) 2009
(c) 2010 (d) 2011

14. A continuous function $y = f(x)$ is defined in a closed interval $[-7, 5]$. $A(-7, -4)$, $B(-2, 6)$, $C(0, 0)$, $D(1, 6)$, $E(5, -6)$ are consecutive points on the graph of f and AB, BC, CD, DE are line segments. The number of real roots of the equation $f[f(x)] = 6$ is

(a) 6 (b) 4 (c) 2 (d) 0

15. Let $f(x) = x^2 + 6x + 1$ and let R denote the set of points (x, y) in the XY-plane such that $f(x) + f(y) \leq 0$ and $f(x) - f(y) \leq 0$. Then the area of the region R is

(a) 6π (b) $3\pi + 2$
(c) $2\pi + 8$ (d) 8π

16. The quadrilateral formed by the lines $y = ax + c$, $y = ax + d$, $y = bx + c$ and $y = bx + d$ has area 18. The quadrilateral formed by the lines $y = ax + c$, $y = ax - d$, $y = bx + c$ and $y = bx - d$ has area 72. If a, b, c, d are positive integers then the least possible value of the sum $a + b + c + d$ is

(a) 13 (b) 14 (c) 15 (d) 16

17. Area of a square $ABCD$ is 36 sq. units and side AB is parallel to the X-axis. Vertices A, B and C lie on the graphs of $y = \log_a x$, $y = 2\log_a x$ and $y = 3\log_a x$ respectively. Then a =

(a) $3^{1/6}$ (b) $\sqrt{3}$ (c) $6^{1/3}$ (d) $\sqrt{6}$

18. The number of three digit numbers with three distinct digits such that one of the digits is the arithmetic mean of the other two is

(a) 120 (b) 180 (c) 112 (d) 104

19. A line is drawn from the point $P(1, 1, 1)$ and perpendicular to a line with direction ratios $(1, 1, 1)$ to intersect the plane $x + 2y + 3z = 4$ at Q. The locus of point Q is

(a) $\frac{x}{1} = \frac{y-5}{-2} = \frac{z+2}{1}$ (b) $\frac{x}{-2} = \frac{y-5}{1} = \frac{z+2}{1}$

(c) $x = y = z$ (d) $\frac{x}{2} = \frac{y}{3} = \frac{z}{5}$

20. The reflection of the curve $xy = 1$ in the line $y = 2x$ is the curve $12x^2 + rxy + sy^2 + t = 0$, then the value of r is

(a) -7 (b) 25 (c) -175 (d) 90

21. Let e be the eccentricity of a hyperbola and $f(e)$ be the eccentricity of its conjugate hyperbola, then $\int_1^3 \underbrace{fff\,..........f(e)}_{n \text{ times}}\, de$ is equal to

(a) $2\sqrt{2}$, if n is odd (b) 4, if n is odd
(c) $2\sqrt{2}$, if n is even (d) 3, if n is even

22. The locus of point of intersection of two tangents to the parabola $y^2 = 4x$ such that their chord of contact subtends a right angle at the vertex is

(a) $x + 4 = 0$ (b) $y + 4 = 0$
(c) $x - 4 = 0$ (d) $y - 4 = 0$

23. If $y, z > 0$ and $y + z = C$, then minimum value of $\sqrt{\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)}$ is equal to

(a) $\frac{C}{2}+1$ (b) $\frac{2}{C}+3$

(c) $1+\frac{2}{C}$ (d) $\frac{C}{2}$

24. $\int \frac{(1+\sqrt{\tan x})(1+\tan^2 x)}{2\tan x}dx$ is equal to

(a) $\log(\sqrt{\tan x})+\sqrt{\tan x}+c$

(b) $\log \tan^2 x+\frac{1}{2\sqrt{\tan x}}+c$

(c) $\log|\tan x|+2\sqrt{\tan x}+c$

(d) $\log|\tan x|+\sqrt{\tan x}+c$

25. If $\left(\frac{3-z_1}{2-z_1}\right)\left(\frac{2-z_2}{3-z_2}\right)=k$, then points $A(z_1)$, $B(z_2)$, $C(3, 0)$ and $D(2, 0)$ (taken in clockwise sense) will

(a) lie on a circle only for $k > 0$
(b) lie on a circle only for $k < 0$
(c) lie on a circle $\forall\ k \in R$
(d) be vertices of a square $\forall\ k \in (0, 1)$

26. Three distinct points A, B and C are given in the two-dimensional coordinate plane such that the ratio of the distance of any one of them from $(2, -1)$ to its distance from $(-1, 5)$ is $1 : 2$. Then the centre of the circle passing through A, B and C is

(a) $(1, 1)$ (b) $(5, -7)$
(c) $(3, -3)$ (d) $(4, -8)$

27. A conic C satisfies the differential equation $(1 + y^2)dx - xydy = 0$ and passes through the point $(1, 0)$. An ellipse E which is confocal with C has its eccentricity $\sqrt{2/3}$. The angle of intersection of the curves C and E is

(a) $\pi/6$ (b) $\pi/4$
(c) $\pi/3$ (d) $\pi/2$

28. $\int \frac{(2+\sec x)\sec x}{(1+2\sec x)^2}dx =$

(a) $\frac{1}{2\operatorname{cosec}x+\cot x}+C$

(b) $2\operatorname{cosec} x + \cot x + C$

(c) $\frac{1}{2\operatorname{cosec}x-\cot x}+C$

(d) $2\operatorname{cosec} x - \cot x + C$

29. The position vectors of the vertices A, B, C of a tetrahedron $ABCD$ are $\hat{i}+\hat{j}+\hat{k}, \hat{i}$ and $3\hat{i}$ respectively and the altitude from the vertex D to the opposite face ABC meets the face at E. If the length of the edge AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, then the length of DE is

(a) 1 (b) 2 (c) 3 (d) 4

30. If a is an integer lying in the closed interval $[-5, 30]$, then the probability that the graph of $y = x^2 + 2(a + 4)x - 5a + 64$ is strictly above the x-axis is

(a) 2/9 (b) 1/6 (c) 1/2 (d) 5/9

31. The number of ways of forming an arrangement of 5 letters from the letters of the word "IITJEE" is

(a) 60 (b) 96 (c) 120 (d) 180

32. The value of p for which the straight lines $\vec{r}=(2\hat{i}+9\hat{j}+13\hat{k})+t(\hat{i}+2\hat{j}+3\hat{k})$ and $\vec{r}=(-3\hat{i}+7\hat{j}+p\hat{k})+s(-\hat{i}+2\hat{j}-3\hat{k})$ are coplanar is

(a) -1 (b) 1 (c) -2 (d) 2

33. If $a_1, a_2, ..., a_n$ are real numbers with $a_n \neq 0$ and $\cos\alpha + i\sin\alpha$ is a root of
$z^n + a_1z^{n-1} + a_2z^{n-2} + ... + a_{n-1}z + a_n = 0$,
then the sum
$a_1\cos\alpha + a_2\cos2\alpha + a_3\cos3\alpha + ... + a_n\cos n\alpha$ is

(a) 0 (b) 1 (c) -1 (d) 1/2

34. If the tangents at z_1, z_2 on the circle $|z - z_0| = r$ intersect at z_3, then $\frac{(z_3-z_1)(z_0-z_2)}{(z_0-z_1)(z_3-z_2)}$ equals

(a) 1 (b) -1 (c) i (d) $-i$

35. Let $f(x) = \int_0^1 |t-x|t\ dt$ for all real x. Then the minimum value of f is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}\left(1+\frac{1}{\sqrt{2}}\right)$

(c) $\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)$ (d) $\frac{1}{6}$

36. If $c > 0$ and the area of the region enclosed by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576, then $c =$

(a) 6 (b) 4
(c) 3 (d) 8

37. Chords of the parabola $y^2 = 4x$ touch the hyperbola $x^2 - y^2 = 1$. The locus of the point of intersection of the tangents drawn to the parabola at the extremities of such chords is

(a) a circle
(b) a parabola
(c) an ellipse
(d) a rectangular hyperbola

38. $\vec{a}$ and $\vec{b}$ are non-zero non-collinear vectors such that $|\vec{a}|=2$, $\vec{a}\cdot\vec{b}=1$ and angle between $\vec{a}$ and $\vec{b}$ is $\pi/3$. If $\vec{r}$ is any vector satisfying $\vec{r}\cdot\vec{a}=2$, $\vec{r}\cdot\vec{b}=8$, $(\vec{r}+2\vec{a}-10\vec{b})\cdot(\vec{a}\times\vec{b})=6$ and is equal to $\vec{r}+2\vec{a}-10\vec{b}=\lambda(\vec{a}\times\vec{b})$ then $\lambda =$

(a) 1/2 (b) 2
(c) $4/\sqrt{3}$ (d) 3

ANSWERS KEYS

1.	(d)	2.	(b)	3.	(d)	4.	(c)	5.	(c)
6.	(d)	7.	(b)	8.	(d)	9.	(c)	10.	(b)
11.	(d)	12.	(b)	13.	(d)	14.	(a)	15.	(d)
16.	(d)	17.	(a)	18.	(c)	19.	(a)	20.	(a)
21.	(a)	22.	(a)	23.	(c)	24.	(a)	25.	(a)
26.	(c)	27.	(d)	28.	(a)	29.	(b)	30.	(a)
31.	(d)	32.	(c)	33.	(c)	34.	(b)	35.	(c)
36.	(a)	37.	(c)	38.	(b)				

PRACTICE PAPER 2015 JEE (Main & Advanced) and other PETs

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19. Match the following:

	Column I	Column II	
(P)	If in a ΔABC, $a = 6$, $b = 3$ and $\cos(A-B) = \frac{4}{5}$, then area of $\Delta ABC =$	1.	0
(Q)	If $a = 5$, $b = 7$, $\sin A = \frac{3}{4}$, then the number of triangles possible is/are	2.	2
(R)	In a ΔABC, $b + c = 3a$, then $\cot\left(\frac{B}{2}\right)\cot\left(\frac{C}{2}\right) =$	3.	3
(S)	$\frac{\{a(b^2+c^2)\cos A + b(c^2+a^2)\cos B + c(a^2+b^2)\cos C\}}{abc} =$	4.	9

	P	Q	R	S
(a)	4	1	2	3
(b)	4	1	3	2
(c)	4	2	3	1
(d)	4	3	2	1

20. If in a triangle $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then match the following :

	Column I	Column II	
(P)	$\cos A$	1.	1/4
(Q)	$\cos B$	2.	7/8
(R)	$\cos C$	3.	0
(S)	$\sin(A-C)$	4.	1

	P	Q	R	S
(a)	1	2	1	4
(b)	2	1	2	3
(c)	1	2	1	3
(d)	1	1	2	3

ANSWERS

PAPER-1

1.	(a, c)			2.	(b, c)				
3.	(a, b)			4.	(a, b)			5.	(a, c)
6.	(a, c)			7.	(b)			8.	(b, c)
9.	(a)			10.	(a, b, c)			11.	(7)
12.	(2)	13.	(6)	14.	(3)	15.	(2)	16.	(6)
17.	(5)	18.	(0)	19.	(9)	20.	(6)		

PAPER - II

1.	(d)	2.	(a)	3.	(b)	4.	(a)	5.	(a)
6.	(d)	7.	(b)	8.	(b)	9.	(c)	10.	(a)
11.	(d)	12.	(c)	13.	(b)	14.	(c)	15.	(d)
16.	(a)	17.	(c)	18.	(b)	19.	(a)	20.	(c)

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